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Rev. T. E. R. PHILLIPS, PRESIDENT, in the Chair.

The election by the Council of the following Fellows was duly confirmed :—

F. C. R. Douglas, M.A., 8 Cambridge Road, Battersea Park, London, S.W. 11 ;

John Leslie Haughton, D.Sc., F.Inst.P., National Physical Laboratory, and 7 Bolton Gardens, Teddington ;

Frederick Neville Hopkins, 31 Raglan Road, Retford, Notts ;

David Perceval Larham, Lieut. R.N.R., 49 Portswood Road, Southampton, Hants ;

C. P. Sivarama Menon, B.A., 112 Gower Street, London, W.C. 1 ;

Dr. Rolf Müller, Astrophysical Observatory, Potsdam, Germany, and Casilla 649, La Paz, Bolivia ;

Philip Augustus Taylor, Ph.D., Mathematical Department, Imperial College of Science and Technology, and 18 Rosenau Crescent, London, S.W. 11 ;

Dr. Heinrich Vogt, Professor of Astronomy, Heidelberg University, and Königstuhl Observatory, Heidelberg, Germany ;

Evan Gwyn Williams, B.A., 71 Eardley Crescent, London, S.W. 5 ; and

C. J. Wilson, Lieut., R.N.R., Mount Faber Observatory, Singapore.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended :—

Arthur T. Doodson, D.Sc., Tidal Institute, University of Liverpool (proposed by J. Proudman) ;

Sir Oliver Joseph Lodge, F.R.S., Normanton House, Lake, Salisbury (proposed by H. H. Turner); and
Neil Henderson Smith, B.A., British Imperial Oil Co., Ltd., Melbourne, Australia (proposed by L. J. Comrie).

One hundred and thirty-five presents were announced as having been received since the last ordinary meeting, including amongst others:—

Barnard (E. E.), A Photographic Atlas of Selected Regions of the Milky Way. Edited by E. B. Frost and M. R. Calvert. 2 vols.
Bauschinger (J.), Die Bahnbestimmung der Himmelskörper. Zweite Auflage, 1928.
Campbell (L.) and W. Garnett, The Life of James Clerk Maxwell (presented by D. Carder-Davies);
Fotheringham (D. R.), The Date of Easter and other Christian Festivals.
Catania Observatory, Astrographic Catalogue, vol. 4, pt. 2; vol. 5, pt. 2.
Helsingfors Observatory, Astrographic Catalogue, tome 1, fasc. 3.

The Deviation of Stellar Material from a Perfect Gas.

By A. S. Eddington, F.R.S.

1. In a gas consisting of charged ions and electrons there is an electrostatic energy (negative) arising from the mutual attractions and repulsions. This gives rise to a pressure amounting to $\frac{1}{3}$ of the electrostatic energy per unit volume, which must be applied as a correction to the pressure as calculated for a perfect gas. Since the correction is negative the gas is superperfect. It is evidently a matter of importance to calculate this correction, or at least to set an upper limit to it, for the conditions corresponding to the interior of a star.

A method of calculation intended primarily for application to electrolytes was given by Debye and Hückel * in 1923. It was applied to stellar material by Rosseland † and by Fowler and Guggenheim. ‡ The latter made numerical calculations giving a value large enough to modify considerably the predicted brightness of stars less massive than the sun. I have shown § that about $\frac{2}{3}$ of this calculated correction is spurious, arising merely from an approximation employed in developing the formulæ. The remaining $\frac{1}{3}$ appeared to be still an overestimate, but in any case the effect as thus reduced would not be of much astro-

* *Phys. Zeits.*, **24**, 185.
† *Monthly Notices*, **84**, 720.
‡ *Ibid.*, **85**, 939.
§ *Ibid.*, **86**, 2.

nomical importance. In 1927 Kramers * developed a new method for electrolytes which gave results larger than those of Debye and Hückel, the difference increasing rapidly for moderately high concentration. Defects in the Debye-Hückel theory have been pointed out by R. H. Fowler, and a paper by J. A. Gaunt in the present number of the *Monthly Notices* gives further light on its invalidity. This raises anew the question whether the Debye-Hückel formula (as corrected in my earlier paper) can still be accepted as setting an upper limit to the astronomical effects.

In my former paper I was a critic of the Debye-Hückel theory and was chiefly concerned to defend my innovations. Now, however, no one seems to have a good word for that theory, and it is the parts of my former calculation which were not dissociated from it which have to be justified or amended. The present investigation follows the Debye-Hückel theory up to a certain point, but deviates on reaching the precarious or invalid part of that theory. The upper limits to the energy reached in the present paper are not greatly changed as compared with those given in the former paper, and I think that my former conclusion holds good, viz. that the deviations from a perfect gas are small and give no sensible correction to the mass-luminosity curve, except that stars of small mass will fall a little above the original curve, the correction (which depends on the assumed chemical constitution) amounting to perhaps half a magnitude for $\frac{1}{4}$ the mass of the sun.

The present investigation must be regarded as a separate theory of the stellar gas; it is not applicable to electrolytes. I think that conversely Kramers' new theory is such that its range of validity (if any) does not include stellar conditions. The question of the degree of importance of the electrostatic pressure is so fundamental in stellar problems that it is undesirable to leave it at the mercy of extraneous controversies, especially as the stellar case appears to be more tractable than that of an electrolyte. I have therefore tried to give the full argument so that it may be scrutinised from the beginning. The principal results assumed from general physics are (1) that the correction to the pressure is $\frac{1}{3}$ the electrostatic energy,† and (2) that the frequency of any configuration is $exp(-W/RT)d\omega$, where $d\omega$ is the elementary range of co-ordinates for classical applications and is to be replaced by the quantum weights when the classical theory is inapplicable.

Before entering on details it is desirable to understand the manner in which deviations from the gas-law occur. The incompressibility of neutral gases at high density is evidently due to some cause which prevents the centres of atoms from approaching too closely. It is a common error to suppose that the essential point is to keep the *nuclei* apart, and that any cause tending to prevent close approaches of the nuclei will diminish the compressibility. But actually this has just the opposite effect; to discourage close approaches of like charges diminishes the electrostatic energy and thereby makes the material more compressible.

* *Proc. Amsterdam Acad.*, **30**, 145.

† See, for example, *Internal Constitution of the Stars*, equation (184·1).

The ordinary atomic volume effect in neutral gas must depend on the fact that when the nuclei are squeezed closer together the electrons are unable to follow them. If we proceeded beyond a certain limit the electrons would begin to be squeezed out—a process requiring prodigious expenditure of energy. This can be seen by considering the extreme case in which the nuclei are packed into one compact mass ; the electrons are not within the mass but surround it in quantum orbits. Ultimately it is the electrons which set a limit to the packing because each has to be provided with a separate quantum orbit, which by Pauli's exclusion principle it cannot share with another.

In ionised gas this effect does not occur. The free electrons roam anywhere (with a preference for places of positive potential) and can be compressed along with the nuclei. The few bound electrons which remain are left plenty of room to describe their separate orbits (except possibly in white dwarf stars, which are not here considered).

In dense media the effect of ionisation is that the electrons can get closer to the nuclei than they could do if bound.

2. Consider an electrically neutral medium consisting of N charged particles (ions and electrons) in a given volume V. We shall consider a configuration a of the N particles, and also a configuration b of $(N - 1)$ particles in the same volume. For definiteness the missing particle is first taken to be an ion (charge Ze). The ensemble of all possible configurations a, *taken in the proportions in which they naturally occur*, will be called A ; B is the corresponding ensemble of b. Of course the frequency with which a configuration b occurs in B is not the same as the frequency with which it occurs (as part of an a) in A.

The frequency of a given b is made up of the $3(N - 1)$ dimensional elementary range of co-ordinates of its particles, denoted by $d\omega'$, multiplied by the Boltzmann factor $e^{-W'/RT}$ where W' is the energy of the configuration. Let ψ be the electrical potential at a point in b. We divide the whole volume V according to the value of ψ so that $v(\psi)d\psi$ is the combined volume of all regions in which the potential is between ψ and $\psi + d\psi$. Further, let $\bar{v}(\psi)$ be the value of $v(\psi)$ averaged over the ensemble B ; so that

$$\bar{v}(\psi)\int e^{-W'/RT}d\omega' = \int v(\psi) \, . \, e^{-W'/RT}d\omega' \quad . \quad . \quad (1)$$

Now take a configuration a of energy W, and mark a particular ion in it. Deleting this ion, a configuration b is left. Let W' be the energy of b, and ψ its potential at the point where the deleted ion was. Then

$$W = W' + Ze\psi \quad . \quad . \quad . \quad . \quad (2)$$

Since $v(\psi)d\psi$ represents an elementary range of the marked ion, the elementary range of a can be expressed as

$$d\omega = d\omega' \, . \, v(\psi)d\psi,$$

and the frequency with which a occurs in A is thus

$$e^{-W/RT}d\omega = v(\psi)e^{-W'/RT}d\omega' \, . \, e^{-Ze\psi/RT}d\psi \quad . \quad . \quad (3)$$

by (2). The number of a's in A corresponding to a particular ψ is obtained by integrating (3) with respect to $d\omega'$; using (1) this gives

$$C\bar{v}(\psi) \, . \, e^{-Ze\psi/RT} d\psi \quad . \quad . \quad . \quad . \quad (4)$$

where

$$C = \int e^{-W'/RT} d\omega'$$

and is a constant independent of ψ.

The formula (4) gives the proportionate number of configurations in A which have a marked ion at potential ψ to $\psi + d\psi$. How is the mark to be interpreted ? It arises from the contemplation of a 3N dimensional region in which the ions are each marked as belonging to three particular co-ordinates, and each ranges over a volume V independently. Any three-dimensional configuration a will be repeated in 3N dimensions with each ion in turn as the marked ion (*i.e.* assigned to the marked set of co-ordinates). Thus if *any* ion in a has potential ψ to $\psi + d\psi$ the configuration will be enumerated in (4) when it is the turn of that ion to be the marked ion ; and the configuration will be counted in (4) as many times proportionately as it has ions at potential ψ to $\psi + d\psi$. Thus (4) is proportional to the total number of ions at this potential in the ensemble A, or to the average number in a natural system. It follows that—

In a natural configuration the average number of ions at potential ψ to $\psi + d\psi$ can be written

$$\sigma_0 \bar{v}(\psi) e^{-Ze\psi/RT} d\psi \quad . \quad . \quad . \quad . \quad (5)$$

Note that ψ has been defined with reference to configuration b ; that is to say, it must be reckoned as though the particular ion were deleted, the other ions being undisturbed.

It is tempting to infer that $\sigma_0 e^{-Ze\psi/RT}$ is the average density of the ions at a place of potential ψ, but it is difficult to say what this would mean. We can take $\bar{v}(\psi)$ to refer to a volume averaged over either A or B, since the change from N to (N − 1) particles will not appreciably affect its functional form. If we refer it to B, the point immediately arises that the volume is averaged over B but the number of ions over A. If we refer it to A, the result of dividing the number of ions by a volume which does not include them can scarcely be described as a density. We therefore leave (5) as a rigorous formula without interpretation in terms of density.

We need not be so scrupulous in interpreting $e^{e\psi/RT}$ as approximately a density of electrons in fields broadly controlled by the more powerful charges of the ions. So long as the removal of the electron under consideration does not greatly alter the extent and locality of the region between ψ and $\psi + d\psi$, the density has an approximate meaning. However, we shall not here need to use this interpretation.

3. From A select the ensemble A_0 consisting of those a's which have an ion at a fixed point O in V. Superpose these and consider their average potential $\bar{\psi}_a$ and average density of electric charge $\bar{\rho}$ at

any point. This average distribution will of course have spherical symmetry about O. The equation

$$\nabla^2\bar{\psi}_a = -4\pi\bar{\rho} \quad . \quad . \quad . \quad . \quad (6)$$

will be rigorously true.

The number of ions distant r to $r + dr$ from O will be proportional to

$$4\pi r^2 dr \int_{0,\,r} e^{-W/RT} d\omega \quad . \quad . \quad . \quad . \quad (7)$$

where the integration is $3(N - 2)$-fold, two of the ions being fixed in infinitesimal ranges of volume at O and r respectively. Transforming as in (3), the part of (7) corresponding to a range $d\psi$ is

$$e^{-Ze\psi/RT} d\psi \int_0 v_r(\psi) e^{-W'/RT} d\omega',$$

where $v_r(\psi)$ is the part of $v(\psi)$ in the shell r to $r + dr$. Omitting constant factors, this reduces as before to

$$\bar{v}_r(\psi) e^{-Ze\psi/RT} d\psi \quad . \quad . \quad . \quad . \quad (8)$$

where $\bar{v}_r(\psi)$ is the average value of $v_r(\psi)$ in the ensemble B_0.

The number of ions in a shell r to $r + dr$, averaged over A_0, is obtained by integrating (8) over all values of ψ. Writing M for " mean value of " we have

$$\int e^{-Ze\psi/RT} \bar{v}_r(\psi) d\psi = M(e^{-Ze\psi/RT}) \int \bar{v}_r(\psi) d\psi = 4\pi r^2 dr \,.\, M(e^{-Ze\psi/RT}).$$

The averaging expressed by M is evidently over the ensemble B_0, and the potential ψ refers to configuration b. But clearly it will make no appreciable difference if we squeeze one extra ion into the large volume V, so that M and ψ may now be taken as referring to A_0 and a. Hence, so far as concerns the ions, the mean charge density is

$$\bar{\rho} = const \times M(e^{-Ze\psi_a/RT}) \quad . \quad . \quad . \quad (9)$$

the averaging being over A_0 for both sides.

In principle equations (6) and (9) constitute the Debye-Hückel theory. It is purely classical, and modifications necessary to take account of the divergence of quantum from classical theory are to be introduced at a later stage. That being understood, I can see no possible doubt as to the validity up to this point.

The next step in the ordinary development is to set

$$M(exp - Ze\psi_a/RT) = exp - Ze\bar{\psi}_a/RT \quad . \quad . \quad (10)$$

in order to have the same variables $\bar{\rho}$, $\bar{\psi}_a$ in both (6) and (9). The legitimacy of neglecting the fluctuations in this way has rightly been criticised.* It appears that in stellar conditions (10) is very far from true, and it is essential to adopt some other procedure.

We shall now show that this difficulty does not arise in the corre-

* I was made aware of the complete failure of (10) in stellar conditions more particularly through the work of J. A. Gaunt. An illustration of the extent of the failure is given in § 12.

sponding discussion of the mean charge density due to the electrons. Our further developments will proceed on the principle of looking after the electrons and letting the ions (literally) do their worst.

4. The theory of §§ 2, 3 rests on the principle that the frequency of a configuration a of a system of particles is proportional to $e^{-W/RT}d\omega$, where W is the potential energy and $d\omega$ is the elementary range of space co-ordinates. This is a simplification of the more general principle of the same form in which W includes kinetic energy also, and $d\omega$ is the 6N dimensional range of both space co-ordinates and velocities. The full derivation of (3) is then

$$W = W' + Ze\psi + \tfrac{1}{2}m(u^2 + v^2 + w^2) \quad . \quad . \quad (11)$$

$$d\omega = d\omega' \, . \, dxdydzdudvdw \quad . \quad . \quad . \quad . \quad (12)$$

and the frequency of systems with the marked ion between potentials ψ and $\psi + d\psi$ becomes

$$\int d\omega' \, . \, v(\psi)d\psi \int exp - \{W' + Ze\psi + \tfrac{1}{2}m(u^2 + v^2 + w^2)\}/RT \, . \, dudvdw$$

$$= C\bar{v}(\psi)d\psi \int\int\int exp - \{Ze\psi + \tfrac{1}{2}m(u^2 + v^2 + w^2)\}/RT \, . \, dudvdw \quad (13)$$

by the same reduction as before. When the integration is over all values of u, v, w the triple integral gives $(\pi m/2RT)^{\frac{3}{2}}exp(-Ze\psi/RT)$, so that the result (4) is obtained.

In seeking the corresponding formula for electrons ($Z = -1$) it is necessary to remember that bound electrons are counted as part of the ion. Since we are concerned only with the distribution of free electrons we must limit the integration to values for which

$$-e\psi + \tfrac{1}{2}m(u^2 + v^2 + w^2) > 0.$$

The result is that in place of $exp(e\psi/RT)$ we have $f(e\psi)/RT)$ where *

$$\left.\begin{aligned} f(e\psi/RT) &= e^{e\psi/RT} \text{ for } \psi \text{ negative} \\ f(x^2) &= \frac{2}{\sqrt{\pi}}\left(x + e^{x^2}\int_x^\infty e^{-t^2}dt\right) \text{ for } \psi \text{ positive} \end{aligned}\right\} \quad . \quad (14)$$

To obtain an idea of the course of this function we set

$$f(e\psi/RT) = 1 + \gamma e\psi/RT \quad . \quad . \quad . \quad (14a)$$

The values of γ are found to be

$e\psi/RT$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	3	4	16
γ	·63	·79	1	·64	·56	·47	·41	·33	·23

The fluctuations are now much less important. The range would in any case be greatly reduced by the fact that we are dealing with $e\psi/RT$ instead of $Ze\psi/RT$. But, further, the function f is sufficiently nearly linear to admit of reasonably accurate averaging, and there is no longer the danger of enormous averaging factors associated with the exponential.

* *Monthly Notices*, **86**, 6.

If we neglect the fluctuations and set (*cf.* (10))

$$M(f(e\psi/RT)) = f(e\bar{\psi}/RT)) \quad . \quad . \quad . \quad (15)$$

we slightly exaggerate the concentration towards places of high positive potential, thereby making our calculation of the negative electrostatic energy an upper limit. Or by taking

$$M(f(e\psi/RT)) = 1 + e\bar{\psi}/RT \quad . \quad . \quad . \quad (15a)$$

we obtain a simple and safe upper limit which is only slightly higher.

5. It is not necessary to draw the line precisely at the zero energy separating free and bound electrons. An electron of energy near zero may be grouped either with the free or with the bound electrons ; but if it is reckoned as a free electron for the purposes of the present calculation of electrostatic pressure, it must also be reckoned as a free electron for the purpose of computing the ordinary gas-pressure ; and, further, it must not be counted as part of the ion in calculating a correction to the gas equation on account of volume of the ion.

For example, consider an electron of total energy $T + U = 0$. If it is counted as a free electron it contributes $\frac{2}{3}T$ to the gas-pressure and $\frac{1}{3}U$ to the electrostatic pressure, giving a balance $\frac{1}{3}T$. If it is counted as a highly excited bound electron, the same contribution to the pressure is attributed to the volume of the excited ion—the b term in Van der Waals' equation. Since the electrostatic pressure is intended to be added to the gas-pressure there must always be an understanding between investigators of each as to where their respective responsibilities end.

It is probably most convenient that the present investigation should take over as far as possible the highly excited electrons, since these are difficult to deal with in investigations of ionisation, and for our own sake we wish to deal with ions as compact as possible. A convenient division would be to treat electrons in orbits above the fourth quantum orbit as free. The corresponding limit of negative energy should then be taken as

$$\chi = 2\pi^2 mZ^2e^4/h^2(4\tfrac{1}{2})^2.$$

The stellar conditions which we shall adopt in § 7 are

$$Z = 24, \qquad T = 2{\cdot}64 \,.\, 10^7 \text{ degrees.}$$

This gives

$$\chi/RT = 0{\cdot}17.$$

We have now to integrate (*cf.* (13))

$$\iiint exp\{e\psi - \tfrac{1}{2}m(u^2 + v^2 + w^2\}/RT \,.\, dudvdw$$

over the range for which

$$\tfrac{1}{2}m(u^2 + v^2 + w^2) > e\psi - \chi.$$

The result is

$$e^{\chi/RT} f((e\psi - \chi)/RT) \quad . \quad . \quad . \quad . \quad (16)$$

where f is the same function as before.

The following table gives a comparison of the densities for different values of $e\psi/RT$, (1) excluding, (2) including electrons excited above the fourth quantum orbit, in the conditions above mentioned ($\chi/RT = 0{\cdot}17$) :

TABLE I.

$e\psi/RT$.	Density (1).	Density (2).	$e\psi/RT$.	Density (1).	Density (2).
0	1·000	1·000	0·68	1·411	1·572
0·17	1·129	1·185	0·85	1·490	1·673
0·34	1·233	1·338	1·02	1·564	1·766
0·51	1·326	1·462	1·19	1·635	1·854

Writing the density in the former manner

$$1 + \gamma e\psi/RT,$$

the inclusion of the highly excited electrons raises the value of γ a little, but except for a very trivial excess near $e\psi/RT = 0{\cdot}17$ it remains below unity and decreases with increasing ψ.

The question may be asked, Why should we not treat all the bound electrons in this manner instead of stopping short of the fourth quantum orbit ? That would save a special investigation of the ionisation. We cannot do this because for close bound electrons our fundamental expression $e^{-W/RT}d\omega$ is invalid. For free particles it is correct both on classical and on quantum theory ; but for bound particles it is only approximate and the approximation fails entirely for low quantum states. The element $d\omega$ is replaced by a " weight " which rapidly approaches the value $dxdydzdudvdw$ for high quantum states, but is altogether different for the first few states.

The question also arises why there is no corresponding modification of the exponential factor for the ions. The first reason is that whereas we do not want to count separately the bound electrons which are merged in the ion, we must count all the ions somehow. Even if the ion has negative energy we should include it in the present investigation, as we have included the highly excited electrons, unless a quantisation occurs of number low enough to make our method inapplicable. Here a second difference becomes important—viz. the great difference of mass of the ions and electrons, which makes the velocities of the ions negligible compared with those of the electrons. We have assumed that electrons with energy below $-0{\cdot}17$ RT will be quantised in low orbits. But energy is relative. The quantisation involves some degree of permanence of controlling conditions, and the energy $-0{\cdot}17$ RT should be reckoned with respect to axes to which the permanent conditions refer ; these are provided by the slow moving ions, and the difference of the electron's energy referred to axes moving with the ion and referred to fixed axes is unimportant. For the heavy ion it makes a great difference what axes the motion is referred to. The only case in which the ions could be affected by quantisation is when they are bound to the points of a crystal lattice. (It is unlikely that the stars are crystalline,

but this possibility will be considered in § 13.) We are thus unable to extend to the ions the simplification which appears in the treatment of electrons.

The case of an electrolyte is altogether different. Here the positive and negative charges are associated with much the same mass, and the treatment of both must be symmetrical. In neither case can it be assumed that because the energy of the particle referred to the usual fixed axes is negative that therefore it is under quantum restrictions. If the negative energy is not due to chance fluctuations in a field due to many ions (in which case the conditions are not permanent enough for quantisation) the controlling field will be travelling with another ion, and the energy referred to that ion will differ by an amount of the order RT from that calculated with respect to fixed axes. Nevertheless for energies less than $-$ RT there is an increasing probability of the ions being bound as a neutral molecule, and it would appear that the exponential is replaced by the harmless f function before it has a chance of introducing very serious fluctuations.

For this reason I think that the Debye-Hückel theory requires no great modification in its application to electrolytes, and that it may be preferable to Kramers' theory, which cannot well be adapted to take account of the binding of ions.

6. An important conclusion follows from § 4, viz. that there can be no tendency for the ions to cluster. Two ions, of course, repel one another strongly, but the suggestion might be made that their combined field would drag in enough electrons to cement them together. If we take a configuration of ions and electrons with negative energy and reduce the linear scale in the ratio l, the electrostatic energy is multiplied in the ratio l. It might seem therefore that the configuration would be tempted by this opportunity of attaining lower (*i.e.* increased negative) energy and break up into small clusters. We see that this cannot happen; because if the ions are pressed closer together the electrons will not follow them. When ψ is increased l-fold the electron-density is increased less than l-fold ($f(lx)<lf(x)$) whereas it would have to increase proportionately to l^3 to conform homologously to the alteration of scale.

Owing to the positive potential there is round each ion an excess of electrons above the number which would exist there if the ion were absent. When two such ions approach the excess is slightly less than the sum of the excesses for each ion separately.* The shielding of the ion's field by its *entourage* of electrons is accordingly diminished rather than increased at close approaches, and the position of least energy is when the ions are as far from one another as possible subject to the assigned limits of volume.

We shall employ this principle by occasionally inserting conditions which exclude close approaches of the ions, the justification being that

* When highly excited electrons are included the superposition gives a very small increase in the outer part of the ionic fields corresponding to $e\psi/\mathrm{RT} = 0{\cdot}17$; but this is more than counterbalanced by the decrease in the stronger parts of the field.

in omitting these configurations we increase the average negative energy for which we are investigating an upper limit.

Clustering of the kind suggested above would evidently signify that the internal pressure in the gas had become negative. We should be warned of such a change of state by finding that the negative electrostatic pressure was approaching equality with the gas-pressure, in investigating a series of conditions of T and ρ. It seems clear, however, that this vanishing of the pressure cannot happen. Unless the ions can concentrate each its own share of electrons in its immediate neighbourhood they will continue to repel one another and spread out as much as possible. If they do effect this concentration they become effectively neutral and their ordinary gas-pressure remains. In short, the pressure in an ionised gas can never be less than the pressure in the corresponding un-ionised gas.

7. As typical of the stellar conditions in which we are interested we take

$$\rho = 156, \qquad T = 2{\cdot}64 \,.\, 10^7, \qquad A = 56, \qquad Z = 24.$$

A being the atomic weight. These are the conditions considered at length in my former paper and correspond to a star of about $\frac{1}{3}$ the sun's mass. The data give $1{\cdot}68 \,.\, 10^{24}$ ions per cu. cm., so that they are spaced about 10^{-8} cm. apart. For the electrons the mean density is

$$\sigma_0 = 4{\cdot}03 \,.\, 10^{25}.$$

Another important linear constant is

$$b = Ze^2/RT = 1{\cdot}51 \,.\, 10^{-9}.$$

Consider the mean field about an ion as in § 3. We make two simplifications, both of which exaggerate the negative electrostatic energy and are therefore permissible in calculating an upper limit: (*a*) we suppose that all other ions are entirely excluded from a sphere of radius R, (*b*) we take $\gamma = 1$ in (14*a*).* So that the charge-density of the mean distribution is

$$\bar{\rho} = -e\sigma_0(1 + e\bar{\psi}/RT).$$

It may be noted that σ_0 is the average density of electrons (given above) as well as the density at zero potential, because the space-average of ψ or $\bar{\psi}$ is zero in an electrically neutral medium. Hence $\bar{\psi}$ is determined by the equation

$$\nabla^2\bar{\psi} = -4\pi\bar{\rho} = \frac{4\pi e^2\sigma_0}{RT}\left(\bar{\psi} + \frac{RT}{e}\right) \quad . \quad . \quad (17)$$

The general symmetrical solution is

$$\bar{\psi} = \frac{Ze}{r}(Ae^{\gamma r} + Be^{-\gamma r}) - \frac{RT}{e} \quad . \quad . \quad . \quad (18)$$

* Including the electrons excited above the fourth quantum orbit, γ falls to 0·79 at $e\psi/RT = 0{\cdot}85$. We are scarcely concerned with its value beyond this point. Thus $\gamma = 1$ gives a good approximation as well as an upper limit.

where ZeA, ZeB are arbitrary constants, and

$$y^2 = 4\pi e^2\sigma_0/RT \quad . \quad . \quad . \quad . \quad (19)$$

The numerical values are

$$1/y = 5{\cdot}60\,.\,10^{-9}\text{ cm.} \qquad 1/by = 3{\cdot}71.$$

For small values of r (18) becomes

$$\bar{\psi} = (A + B)Ze/r + (A - B)Zey - RT/e \quad . \quad . \quad (20)$$

so that for a charge Ze at the origin we must have $A + B = 1$.

The potential at the origin excluding that of the ion itself is

$$(A - B)Zey - RT/e = Zey(A - B - 1/by) \quad . \quad . \quad (21)$$

and the energy of the ion * is then

$$\tfrac{1}{2}Z^2e^2y(A - B - 1/by) \quad . \quad . \quad . \quad (22)$$

The charge within a sphere of radius r is

$$-r^2d\bar{\psi}/dr = Ze\{Ae^{yr}(1 - yr) + Be^{-yr}(1 + yr)\},$$

so that the number of electrons in the sphere is

$$Z\{1 - Ae^{yr}(1 - yr) - Be^{-yr}(1 + yr)\}$$
$$= Z(\tfrac{1}{2}y^2r^2 + \tfrac{1}{3}(A - B)y^3r^3 + \tfrac{1}{8}y^4r^4 + \ldots) \quad (23)$$

We can also find the radius r' of a sphere within which the average potential is zero; this is given by

$$A(te^t - e^t + 1) + B(-te^{-t} - e^{-t} + 1) = \tfrac{1}{3}t^3/by,$$

where $t = yr'$.

In Table II. we have tabulated for different values of A the distance r_0 at which $\bar{\psi}$ vanishes, the number of electrons n within the sphere r_0, the number $n_0 = \frac{4}{3}\pi r^3\sigma_0$ which would have been within the sphere if there had been uniform density, and a few values of r'.

TABLE II.

A.	B.	yr_0.	r_0.	n.	n_0.	$n-n_0$.	r'.
0·00	1·00	0·215	$1{\cdot}215.10^{-9}$	0·48	0·30	0·18	$1{\cdot}92.10^{-9}$
0·99	0·01	·400	2·24	2·51	1·90	0·61	3·39
1·20	−0·20	·500	2·80	4·62	3·71	0·91	4·37
1·317	−0·317	·600	3·36	7·66	6·40	1·26	5·52
1·384	−0·384	·700	3·92	11·70	10·17	1·53	..
1·418	−0·418	·800	4·48	16·95	15·18	1·77	..
1·428	−0·428	·900	5·04	23·52	21·60	1·92	..

The column $n - n_0$ shows that the excess of electrons in the neighbourhood of an ion is small. Beyond r_0 the mean potential is

* We define the energy of a charge Ze as $\frac{1}{2}Ze\psi$, so that the sum of this expression over all charges, positive and negative, gives the total energy of the material.

negative, so that there is a deficit. By (23) we can locate the excess more definitely. Thus, taking the case $yr_0 = 0{\cdot}7$, we find the excess of $1\frac{1}{2}$ electrons is distributed as follows :—

$0{\cdot}52$ electrons between $r = 0$ and $1{\cdot}4 \cdot 10^{-9}$,
$0{\cdot}50$,, ,, $r = 1{\cdot}4$,, $2{\cdot}24 \cdot 10^{-9}$,
$0{\cdot}51$,, ,, $r = 2{\cdot}24$,, $3{\cdot}92 \cdot 10^{-9}$.

The values of $e\psi/\mathrm{RT}$ at these limits are $0{\cdot}59$, $0{\cdot}22$, 0.

The excess electrons are thus rather close to the ion and for many purposes we can treat them as bound—effectively reducing Z by about $1\frac{1}{2}$. The residual electron density will be nearly uniform.

It will be seen from Table I. that the bound (excited) electrons constitute rather less than $\frac{1}{3}$ of the excess at these potentials. Thus the number of electrons excited above the fourth quantum orbit is slightly under $\frac{1}{2}$ an electron per ion. Each of them will contribute RT to the gas-pressure and $-\frac{1}{3}e\psi$ to the electrostatic pressure, making a total of about $0{\cdot}88$ RT for the mean value of ψ concerned. If these electrons were treated as bound the resulting pressure of $0{\cdot}44$ RT per ion would be called the "excluded volume correction." Our result is a little less than the value usually given ($\frac{1}{2}$ to $\frac{2}{3}$ RT per ion), but we have expressly excluded excitation in the fourth and lower orbits.

8. By (22) the average energy $-$ U of an ion is given by

$$\mathrm{U} = \tfrac{1}{2}\mathrm{Z}^2e^2y\{3{\cdot}71 - (\mathrm{A} - \mathrm{B})\} \quad . \quad . \quad . \quad (24)$$

Thus U decreases as we go down Table II. The problem is to discover whereabouts in the table the actual conditions are to be found, *i.e.* to fix the constant A $-$ B.

It seems evident that the earlier entries represent an impossible state of affairs. Thus in the second line the ion is surrounded by zero potential at the small distance of $2{\cdot}24 \cdot 10^{-9}$ cm. ; the ions only dominate about a twelfth of the whole volume, the remainder having actually on the average the opposite potential sign. In picturing these ions as peaks of positive potential almost submerged in valleys of negative potential, we must not forget that the ions seek or strive to create a site in as deep a valley as possible. But it is difficult to believe that any kind of arrangement of the ions can (not in one case only but generally) cause deep enough hollows to hide them like this.

Our natural expectation is that $n_0 = 12$, $yr_0 = {\cdot}75$ will be near the truth ; because this gives half of the whole volume inside the spheres of zero potential. (The fraction of the volume can be ascertained from the table by dividing n_0 by 24.) Since in a neutral medium the average potential throughout space is zero, this is a likely arrangement. There is in fact reason to believe that $\bar{v}(\psi)$ is distributed according to the error law (see § 12) ; if so, the positive and negative volumes will be equal. But here the difficulty arises that the sphere r_0 is an average. For the individual ion the zero equipotential will be an irregular surface (perhaps unclosed), and there is no guarantee that

the volume of the sphere r_0 will have the average volume of the individual enclosures.*

Although it is difficult to prove that the fluctuations compensate, general considerations show that they are likely to do so approximately, at least for the smaller spheres. The mere fact that the ion tries to find the deepest point of the valley, or creates it by symmetrical repulsions, suggests that the nearer contours will be more or less spherical; at any rate there is not likely to be systematic irregularity which might invalidate the averaging. Further, the ion itself prevents the close approach of other ions, so that the fluctuations are much smaller than those which might occur at an arbitrary point in the medium.

It is not of great importance to fix n_0 very accurately. Thus taking the fourth and last lines of the table:

$$\text{for } n_0 = 6{\cdot}4 \text{ average energy of ion} = -\tfrac{1}{2}Z^2e^2y(2{\cdot}08),$$
$$\text{,, } n_0 = 21{\cdot}6 \quad \text{,,} \qquad \text{,,} \qquad = -\tfrac{1}{2}Ze^2y\ (1{\cdot}86).$$

We should like to be sure that n_0 is not less than the first of these values. Some such limit must be found because the table has no natural limit in the upper direction; we might have extended it to negative values of A.

9. Let us now arrange the ions in the way which gives the highest negative energy. The arrangement will be some form of crystal lattice. The theory of § 7 makes no assumption as to the arrangement of the ions except that there is a minimum distance R within which they do not approach one another. Accordingly the solution will be valid up to the distance R equal to the minimum distance between two points of the lattice; for the face-centred cubic lattice $R = 9{\cdot}44 \cdot 10^{-9}$. The equipotentials immediately surrounding each ion will now be polygonised spheres, which as the potential diminishes may become stellate and shoot out connecting channels. So long as they remain detached there is not much error in averaging them into spheres. When they join up the problem is more complicated, but it is still possible to make estimates sufficiently close for our purpose. By proceeding to this limiting case of a crystal, fluctuations are abolished altogether and the averaging is merely a reduction of symmetrical but non-spherical contours to spheres.

Suppose for trial that $yr_0 = 0{\cdot}5$. We surround the ions with spheres of radius r_0 which will enclose about $\frac{1}{6}$ of the whole volume. Since the average potential on any one of these spheres is zero, the zero equipotential must intersect it; and owing to the symmetry of the lattice the intersections will be distributed in a symmetrical way over the sphere. It becomes evident that an equipotential drawn to make these intersections will leave much more than half the volume on the negative side, making it impossible to satisfy the condition that the average potential through the volume should be zero. Hence $yr_0 > 0{\cdot}5$. We can probably deduce in the same way that $yr_0 > 0{\cdot}6$ without much risk.

* An illustration of this is afforded by the Debye-Hückel formula for the average potential at a distance r from an ion, viz. e^{-yr}/r. This makes r_0 infinite, although there may well be as much negative as positive volume.

If instead of seeking safe limits we try to make the closest estimate we can, we may use $yr_0 = 0{\cdot}75$, which makes the spheres enclose half the volume; they do not overlap. Presumably the actual zero equipotential is a closed convex surface surrounding the centre of each tetrahedron in the lattice (possibly breaking through a face of the tetrahedron to join its neighbour). The negative space will encroach on the spheres r_0 to some extent, and there will be some positive space outside them forming channels along the edges of the tetrahedron. The condition that the average potential is zero over r_0 gives a general idea how to picture these encroachments. It looks as though they would nearly balance; but perhaps the positive channels are likely to exceed the negative encroachments, and the value $yr_0 = 0{\cdot}7$ is preferable.

I conclude therefore that

$$yr_0 = 0{\cdot}7 \qquad \text{energy} = -\tfrac{1}{2}Z^2e^2y(1{\cdot}94)$$

is the likely value, and

$$yr_0 = 0{\cdot}5 \qquad \text{energy} = -\tfrac{1}{2}Z^2e^2y(2{\cdot}31)$$

is a very safe limit.

In my previous discussion the average energy of the ion was *

$$-\tfrac{1}{2}Z^2e^2y(1{\cdot}55).$$

Some increase on the limit was inevitable since we give up the former (invalid) attempt to find the distribution of the ions, being content to place them in the most unfavourable way.

10. Dr. L. H. Thomas has given me a calculation by Ewald's method of the energy of a face-centred cubic lattice, the negative charge being distributed with uniform density. The dimensions are defined by a length d such that the volume d^3 contains 4 ions. The energy per ion is

$$-4{\cdot}585(\tfrac{1}{2}Z^2e^2/d).$$

For our data

$$d = 1{\cdot}335 \,.\, 10^{-8}, \qquad yd = 2{\cdot}385,$$

so that the energy is

$$-\tfrac{1}{2}Z^2e^2y(1{\cdot}923).$$

Two corrections must be applied. If $yr_0 = 0{\cdot}7$ the 1·53 electrons concentrated near each ion reduce its effective charge for great distances from 24 to 22·47, and Thomas's result must be reduced in this ratio. This reduces the coefficient to 1·800. But we must add the energy due to the potential at the ion caused by its own 1·53 electrons. The potential at the ion (excluding itself) is

$$-Zey(3{\cdot}71 - \mathrm{A} + \mathrm{B}) = -1{\cdot}942Zey.$$

The potential at r_0, excluding that of the ion, is

$$-Ze/r_0 = -1{\cdot}429Zey.$$

* Owing to a change of nomenclature, the present y is $(Z+1)^{\frac{1}{3}}$, *i.e.* 5 times, the quantity denoted by y in the earlier paper.

Of the difference ·513 *Zey* the crystal calculation includes $\frac{1}{2}E/r_0$ where E is the assumed uniform negative charge in the sphere. We find E = 9·52 and the corresponding potential is ·283*Zey*, leaving ·230*Zey* due to excess electrons. The revised energy per ion is thus

$$-\tfrac{1}{2}Z^2e^2y(2\cdot030).$$

If we had calculated the correction on the basis $yr_0 = 0\cdot6$ the revised coefficient would have been 1·921.

Thus the calculation of the crystal energy closely confirms our estimates in § 9.

11. To the energy of the ions it is necessary to add the energy $\frac{1}{2}\Sigma e\psi$ for the electrons. If the electrons were uniformly distributed through space this would vanish. Subtracting a uniform distribution σ_0, the energy is (for $\gamma = 1$) the integral of $\frac{1}{2}\sigma_0 e^2\psi^2/RT$ throughout the volume. We shall calculate it for the part of space within r_0 and double the result to allow for the region of negative ψ. Since the amount is small a rough estimate is sufficient.

By (18) this gives (per ion)

$$\left.\begin{aligned} &-\frac{4\pi\sigma_0e^2}{RT}Z^2e^2\int_0^{r_0}(Ae^{yr}+Be^{-yr}-r/b)^2dr \\ &=-Z^2e^2y\int_0^{yr_0}(Ae^{t}+Be^{-t}-t/by)^2dt \\ &=-\tfrac{1}{2}Z^2e^2y(0\cdot186) \end{aligned}\right\}\ \text{by (19)}$$

for $yr_0 = 0\cdot7$.

Adding this to the result in § 9 we may take the whole electrostatic energy of the gas to be

$$-U = -\tfrac{1}{2}Z^2e^2y(2\cdot1)\ \text{per ion.}$$

The ratio of electrostatic pressure to gas-pressure is

$$\left.\begin{aligned} -\Delta p/p &= \tfrac{1}{3}U/(Z+1)RT \\ &= 0\cdot35\ byZ/(Z+1) \\ &= \cdot091 \end{aligned}\right\}\quad . \quad . \quad . \quad (25)$$

It seems natural to associate this reduction of pressure by 9·1 per cent. with the immobilisation of 10 per cent. of the particles. Out of 25 particles 1 (an ion) has been assumed to be fixed at a lattice point, and 1·5 electrons are held as the excess in that neighbourhood, leaving $22\frac{1}{2}$ free to exert the ordinary pressure. This, of course, is not the actual *modus operandi*, but I believe that the equivalence of effect is not a mere coincidence. If we draw a plane section through the gas the pressure across the plane is reduced by the cohesive electrical forces across it. Wherever the cohesive force is strong additional particles must be placed to maintain equilibrium. By omitting at the same time the cohesive forces and the immobilised electron density which acts as buttress against it we ought to obtain nearly the right value of the pressure, as in fact we do.

If this point of view is admissible, our upper limit of 9·1 per cent. cannot be a great overestimate. Sixty per cent. of it comes from the excess electrons. The remainder is from the ions which are actually not so immobile as we have supposed ; but they hold each other apart in a way which amounts to a great restriction on their freedom.

12. *Fluctuations.*—It is probable that $v(\psi)$ is very approximately an error-function, except for large values of ψ. Deviation from the error-law will begin when we reach values which only occur in close proximity to an ion or electron whose contribution to the potential then outweighs everything else. This conclusion rests on the fact that anything to which a considerable number of independent causes contribute linearly will follow the error-law whatever may be the law of fluctuation of the individual causes, subject to the proviso that the composite contribution is not outweighed by one or two preponderating influences. Our result would follow (as in the usual proof of the error-law) if we might suppose that the charges contributing to ψ had small but otherwise arbitrary independent fluctuations of position. Actually the fluctuations are correlated by the occurrence of the factor $e^{-W/RT}$ in the expression for their probability ; it is only through this factor (which prevents charges of like sign from congregating) that the fluctuation of ψ is kept finite. The substitution of correlated for independent causes does not necessarily invalidate the proof, because by a linear transformation we can find uncorrelated variables on which ψ also will depend linearly. For a strict proof it would be necessary to discover this transformation in order to find out whether there is a danger that one or two of the independent contributions will outweigh the majority.

An error-function distribution for ψ seems to follow immediately from Kramers' investigation (*loc. cit.*). He makes the ensemble depend on variables y which are distributed independently according to the error-law. The potential at a point is a linear function of the charges which in turn are linear functions of the y's. Hence ψ also has an error-function distribution. Kramers' investigation is, however, not very convincing, and seems to skirt round the difficult point. But I think it is very unlikely that the conclusion would be upset by any unexamined contingency.

Let us accordingly take

$$\bar{v}(\psi) = \frac{V\alpha}{\sqrt{\pi}} e^{-\alpha^2\psi^2} \quad . \quad . \quad . \quad . \quad (26)$$

so that $1/\alpha\sqrt{2}$ is the standard deviation of ψ. Write $Ze/RT = \beta$. Then the average potential of the positive ions is, by (5),

$$\psi_p = \int_{-\infty}^{\infty} \psi e^{-\alpha^2\psi^2-\beta\psi} d\psi \div \int_{-\infty}^{\infty} e^{-\alpha^2\psi^2-\beta\psi} d\psi = -\beta/2\alpha^2 . \quad (27)$$

Thus the average energy of an ion is

$$-Z^2e^2/4\alpha^2RT.$$

Similarly the average energy of an electron is $-e^2/4\alpha^2RT$; since there

are Z electrons to each ion the energy of the whole material per ion is accordingly

$$-U = -Z(Z+1)e^2/4a^2RT.$$

Thus

$$-\frac{\Delta p}{p} = \frac{\frac{1}{3}U}{(Z+1)RT} = \frac{Ze^2}{12a^2R^2T^2} = \frac{1}{Z}\frac{\beta^2}{12a^2} \quad . \quad (28)$$

which enables us to determine a. Thus for $\Delta p/p = -\frac{1}{10}$, we find the standard deviation of $e\psi/RT$ is $\pm$ 0·16.

We now examine the fluctuations in the Debye-Hückel equation (10). Here ψ has a mean value $\bar{\psi}$, and we shall suppose the standard deviation to be $1/a_r\sqrt{2}$. Then

$$M(exp - \beta\psi) = \int_{-\infty}^{\infty} exp - \{a_r^2(\psi - \bar{\psi})^2 + \beta\psi\}d\psi \div \int_{-\infty}^{\infty} exp - a_r^2(\psi - \bar{\psi})^2 d\psi$$

$$= exp(\beta^2/4a_r^2). \quad exp - \beta\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Thus equation (10) is in error by an averaging factor $exp(\beta^2/4a_r^2)$.

If $a_r = a$ the factor is, by (28),

$$exp(-3Z\Delta p/p) \quad . \quad . \quad . \quad . \quad (30)$$

Thus if the electrostatic pressure is no more than 2 per cent. the factor is nearly $e^{1\cdot5} = 4\cdot5$, and for a pressure of 10 per cent., which is beginning to have astronomical importance, the factor is 1800. Thus equation (10) cannot be applied except in cases of no astronomical interest.

If a_r were independent of r this factor would do no harm, being a mere constant to be absorbed with other constants; but it is easily seen that it increases with r. For large values of r the spherical shell will include a fair average sample of the volume, so that $a_r = a$; for small values the ion at the origin will keep away other ions and reduce the chance fluctuations of ψ, so that $a_r > a$. In (29) the variation of a_r with r is likely to be more important than the variation of $\bar{\psi}$.

The fact is that the probability of occurrence of these highly charged ions is almost negligible except at places of unusually low potential, and a calculation of the average potential has very little relevance. Although we formally regard the error of the Debye-Hückel theory as consisting in the failure of equation (10), it probably originated in a confusion of configurations a and b. If we remove an ion from a normal configuration a the residual configuration b is one of great improbability; therefore it is not much use to consider average potential of the *normal* configurations b.

13. The foregoing discussion naturally suggests the question, Will the stellar gas crystallise? It is not contemplated that the whole star would form a crystal, but single elements or possibly alloys might form crystals which would no doubt have a certain optimum size.

With the data that we have been discussing the negative potential energy of the ion at its lattice point is about 4 times its average kinetic energy. This is scarcely sufficient to hold it there. But the conditions

would presumably be more favourable for heavier elements than we have been considering. A great deal depends on the amount of libration admissible without breaking up the crystal, and whether the total escape of one ion from its lattice point would cause the structure to collapse. I am inclined to think that the possibility of gaseous crystals is very dubious under stellar and perhaps under any conditions.

The crystals here suggested would be genuinely gaseous—not far from perfect gas. The ions form the lattice and the electrons moving almost freely create an ordinary gas-pressure. The crystal tends to expand like a gas and can only exist under pressure.

It is to be noticed that we have only been able to tackle the problem in this way because of the great disparity in charge between the ions and electrons. The method would not apply to a hydrogen star. On the other hand, it is because such disparity is likely that the astronomer needs to investigate the problem. If Z is small the electrostatic pressure cannot be of astronomical importance. This is one of the reasons why Kramers' theory is inappropriate to astronomical problems, for it assumes a symmetrical distribution of positive and negative charges.

Although in the typical star chosen the upper limit to the pressure is increased about 30 per cent. compared with my previous investigation, the former results for the correction to the mass-luminosity curve may be taken as typical. There is no special sanctity about the number $Z = 24$, and the correction will go up or down according to our view as to the chemical constitution of the interior. We can only conclude as before that the correction will brighten up stars of mass less than half that of the sun to an extent which is more or less on the verge of astronomical importance.

The Debye-Hückel Theory and Stellar Atmospheres. By J. A. Gaunt.

(*Communicated by Professor A. S. Eddington.*)

1. *Introduction.*

The present communication falls into two parts. The first is supplementary to a recent criticism of the Debye-Hückel theory by Mr. R. H. Fowler,* using fundamental principles of statistical mechanics. It points out an additional source of error, whose effect seems to be incalculable.

The second part summarises an abortive attempt to distinguish between bound and free electrons in an ionised atmosphere in the manner suggested by Professor Eddington.† For this purpose the differential equations of the theory (as modified by Eddington) must be solved with greater accuracy than is customary. By use of the

* *Trans. Faraday Soc.*, **23**, 411, 1927.

† *The Internal Constitution of the Stars* (Camb. Univ. Press, 1926), p. 262.

Green's functions of the equations,* the second approximation is reduced to quadratures. The integrals, however, are unpleasantly complicated; and the only approximation which I have been able to invent is not valid in those cases which are of interest in stellar physics.

2. *The Foundations of the Debye-Hückel Theory.*

The method and (with a few exceptions) the notation of this section are those of Fowler's paper referred to above. The chief difference in notation consists in the distinction between two kinds of mean density of charge which have previously been confused, and which it is the object of this section to disentangle.

The mean density in the neighbourhood of an ion α is denoted by ρ_α. The words "in the neighbourhood of α" indicate that in the averaging α remains fixed, and that ρ_α is a function of distance from α. Thus

$$\rho_\alpha(x-x_\alpha)\int_{(V)}\ldots\int e^{-W/kT}\Pi_\kappa'(dw_\kappa)^{N_\kappa} = \int_{(V)}\ldots\int \rho(x)e^{-W/kT}\Pi_\kappa'(dw_\kappa)^{N_\kappa} \quad . \quad (2.00)$$

There is another kind of mean density, related to two ions α and β. It is the mean of the density at the position of β (whose charge, however, is not counted), when β as well as α is fixed during the averaging. It is denoted by $\rho_{\alpha\beta}$.

$$\rho_{\alpha\beta}(x_\beta-x_\alpha)\int_{(V)}\ldots\int e^{-W/kT}\Pi_\kappa''(dw_\kappa)^{N_\kappa} = \int_{(V)}\ldots\int \rho'(x_\beta)e^{-W/kT}\Pi_\kappa''(dw_\kappa)^{N_\kappa} \quad (2.01)$$

where Π'' indicates that both dw_α and dw_β are omitted from the product, and ρ' indicates that the charge on β itself is not counted. Were there no smoothing, ρ' and so $\rho_{\alpha\beta}$ would vanish.

The electrostatic potential in any configuration is denoted by ψ; its mean in the neighbourhood of α by ψ_α. A dash denotes the omission from ψ_α of the potential due to α itself.

$$\psi_\alpha(x)\int_{(V)}\ldots\int e^{-W/kT}\Pi_\kappa'(dw_\kappa)^{N_\kappa} = \int_{(V)}\ldots\int \psi(x)e^{-W/kT}\Pi_\kappa'(dw_\kappa)^{N_\kappa} \quad . \quad (2.02)$$

$$\psi_\alpha' = \psi_\alpha - z_\alpha\epsilon/r. \quad . \quad . \quad . \quad . \quad (2.03)$$

The mean mutual potential energy, $W_{\alpha\beta}$, of two ions is so defined by Fowler that the mean density in the neighbourhood of α is

$$\rho_\alpha = \sum_\gamma z_\gamma\epsilon\frac{N_\gamma}{V}e^{-W_{\alpha\gamma}/kT} \quad . \quad . \quad . \quad (2.10)$$

The assumption of Debye and Hückel which is open to question is that $W_{\alpha\beta} = z_\beta\epsilon\psi_\alpha$. Now $W_{\alpha\beta}$ satisfies the equation

$$\frac{\partial W_{\alpha\beta}}{\partial x_\beta}\int_{(V)}\ldots\int e^{-W/kT}\Pi_\kappa''(dw_\kappa)^{N_\kappa} = \int_{(V)}\ldots\int \frac{\partial W}{\partial x_\beta}e^{-W/kT}\Pi_\kappa''(dw_\kappa)^{N_\kappa} \quad . \quad (2.11)$$

by differentiation and addition of similar equations.

* *Cf.* Gronwall, *Proc. Nat. Acad. Sci.*, April 1927, p. 198.

$$\left\{ \nabla_\beta{}^2 W_{\alpha\beta} - \frac{1}{kT}(\nabla_\beta W_{\alpha\beta})^2 \right\} \int_{(V)} \cdots \int e^{-W/kT} \Pi_\kappa{}'' (dw_\kappa)^{N_\kappa}$$

$$= \int_{(V)} \cdots \int \left\{ \nabla_\beta{}^2 W - \frac{1}{kT}(\nabla_\beta W)^2 \right\} \Pi_\kappa{}'' (dw_\kappa)^{N_\kappa} \qquad (2.12)$$

Now $\nabla_\beta W = z_{\beta\epsilon} \nabla_\beta \psi'(x_\beta)$, where the dash denotes the omission of the potential due to β itself. Thus

$$\nabla_\beta{}^2 W = z_{\beta\epsilon} \nabla_\beta{}^2 \psi'(x_\beta) = -4\pi z_{\beta\epsilon} \rho'(x_\beta) \quad . \qquad (2.13)$$

by Poisson's equation.* So, using (2.01)

$$\nabla_\beta{}^2 W_{\alpha\beta} = -4\pi z_{\beta\epsilon} \rho_{\alpha\beta} + \frac{1}{kT}\left\{ (\nabla_\beta W_{\alpha\beta})^2 - \overline{(\nabla_\beta W)^2} \right\} \quad . \qquad (2.14)$$

where the bar denotes a mean with α and β fixed. In the same sense (2.11) gives $\nabla_\beta W_{\alpha\beta} = \overline{\nabla_\beta W}$; so that the last term of (2.14) is a fluctuation term.

A similar equation for ψ_α is obtained by differentiating (2.02), and using Poisson's equation and (2.00). It is

$$\nabla^2 \psi_\alpha = -4\pi \rho_\alpha \quad . \qquad (2.15)$$

Thus the differential equations for $W_{\alpha\beta}$ and $z_{\beta\epsilon}\psi_\alpha$ differ in two respects. There is the fluctuation term in (2.14) which has been studied by Fowler; and there is the distinction between $\rho_{\alpha\beta}$ and ρ_α which must now be discussed.

$\rho_{\alpha\beta}$ differs from ρ_α because in its determination β is fixed at the point at which the density is being measured. As far as short-range forces are concerned, β tends to repel all other ions and so to reduce the numerical value of $\rho_{\alpha\beta}$. If the smoothing is sufficient to wipe out this effect, the electric field of β is still important. To smooth it away would be to destroy the whole theory, which is primarily concerned with such local fields. Since β repels like charges and attracts unlike charges, $\rho_{\alpha\beta} < \rho_\alpha$ if the charge on β is positive, and $> \rho_\alpha$ if it is negative. That is to say,

$$-4\pi z_{\beta\epsilon} \rho_{\alpha\beta} > -4\pi z_{\beta\epsilon} \rho_\alpha \quad . \qquad (2.20)$$

Since the last term of (2.14) is necessarily negative, the two differences from (2.15) tend to cancel each other.

The boundary conditions for $W_{\alpha\beta}$ and $z_{\beta\epsilon}\psi_\alpha$ are the same. Both vanish at infinity. If there is a certain region round α within which another ion never (or very rarely) enters, then $\nabla\psi$ is the same in this region for all (or nearly all) configurations; so that by (2.02) $\nabla\psi_\alpha$ has this value, and by (2.11) $\nabla_\beta W_{\alpha\beta}$ has the same value multiplied by $z_{\beta\epsilon}$. Near α and for low densities both $W_{\alpha\beta}$ and $z_{\beta\epsilon}\psi_\alpha$ consist chiefly of the term $z_\alpha z_\beta \epsilon^2 / r$. Their differences from this term are likely to be of the same order of magnitude, and (2.20) suggests some hope that they may

* With unit dielectric constant, as in stellar atmospheres.

be nearly equal. This, however, is about all that can be deduced from (2.14) in the absence of any estimate of $\rho_{\alpha\beta}$. Unfortunately the details of $W_{\alpha\beta}$ are important.

This is not because $W_{\alpha\beta}$ is used directly in calculating the work function. This function is obtained by integrating

$$\left.\begin{aligned} dA &= \frac{d\sigma}{B_\sigma(T)}\int_{(V)}\ldots\int We^{-\sigma W/kT}\Pi_\kappa(dw_\kappa)^{N_\kappa} \\ &= \sum_\alpha \frac{N_\alpha V d\sigma}{B_\sigma(T)}\int_{(V)}\ldots\int \tfrac{1}{2}z_\alpha\epsilon\psi'(x_\alpha)e^{-\sigma W/kT}\Pi_\kappa'(dw_\kappa)^{N_\kappa} \\ &= \sum_\alpha N_\alpha z_\alpha\epsilon\, dt\,\psi_\alpha'(t) \end{aligned}\right\} \quad (2.21)$$

where $\psi_\alpha'(t)$ is the result of reducing every charge by the factor t in ψ_α'. It is thus the details of ψ_α which are primarily required. But (2.15), (2.10) give, with the usual approximation,

$$\nabla^2\psi_\alpha = 4\pi\sum_\gamma z_\gamma\epsilon\frac{N_\gamma}{V}\frac{W_{\alpha\gamma}}{kT} \quad . \quad . \quad . \quad (2.22)$$

Suppose we try $W_{\alpha\gamma} = z_\alpha z_\gamma\epsilon^2/r$, then the solution of (2.22) contains a term which $\propto r$, and so $\to\infty$ at great distances. This is obviously wrong; but the removal of the infinity depends on the details of $W_{\alpha\gamma}$. A sufficient justification for the replacement of (2.22) by the usual equation would be that $W_{\alpha\gamma}$ and $z_\gamma\epsilon\psi_\alpha$ should differ by only a small percentage at all distances. This percentage is reckoned on the whole of $z_\gamma\epsilon\psi_\alpha$ and not on $z_\gamma\epsilon\psi_\alpha'$, which is an advantage at small distances from α; but it must also be a small percentage at large distances where ψ_α is very small—numerically smaller than ψ_α'.

The above discussion is inconclusive. It seems to show that statistical mechanics provides no firm basis for Debye and Hückel's use of Boltzmann's theorem on an atomic scale. On the other hand, it does not contradict the Debye-Hückel theory, but shows some indications in its favour. That the results of the theory are correct for low densities has been recently proved by Kramers * by a very different method.

3. *An Attempt upon the Stellar Problem.*

We consider an ionised atmosphere in thermodynamic equilibrium, having volume V and temperature T, and containing N ions of charge $z\epsilon$, and zN free electrons of charge $-\epsilon$. Modifications to suit the case of many kinds of ions are mentioned later. The effective radii of the ions prove to be negligible when they are given Eddington's value of 2.10^{-10} cm.† They are neglected in this summary.

Ψ, ψ denote the mean electrostatic potential in the neighbourhoods of an ion and an electron respectively; r is distance from the particle

* *Amsterdam Proc.*, **30**, 1.

† *The Internal Constitution of the Stars*, p. 166.

in whose neighbourhood the field is being found ; κ is the usual constant given by

$$\kappa^2 = \frac{4\pi\epsilon^2}{kT}\frac{zN}{V}(z+1) \quad . \quad . \quad . \quad (3.00)$$

also

$$x = \kappa r, \qquad Y = \epsilon\Psi/kT, \qquad y = -\epsilon\psi/kT, \qquad b = \epsilon^2\kappa/kT \quad (3.01)$$

As explained by Eddington, the density of free electrons in the neighbourhood of an ion is

$$\frac{zN}{V}f(\epsilon\Psi/kT) \quad . \quad . \quad . \quad . \quad (3.10)$$

where

$$f(Y) = \frac{2}{\sqrt{\pi}}\int_Y^\infty e^{Y-x}x^{\frac{1}{2}}dx \quad . \quad . \quad . \quad (3.11)$$

Reciprocally, the density of ions in the neighbourhood of a free electron must be

$$\frac{N}{V}f(\epsilon\Psi/kT) \quad . \quad . \quad . \quad . \quad (3.12)$$

The incomplete symmetry of (3.10) and (3.12) is due to neglect of the velocity of the ion in obtaining the former. In comparison with a fast-moving electron, the ion may be treated as stationary.

Using (3.00), (3.01), the equations in the neighbourhoods of an ion and an electron are :

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{dY}{dx}\right) = -\frac{1}{z+1}\{e^{-zY} - f(Y)\} \quad . \quad . \quad (3.20)$$

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) = -\frac{1}{z+1}\{e^{-y} - f(Y)\} \quad . \quad . \quad (3.21)$$

with the boundary conditions

$$\left.\begin{array}{c} Y,\ y \to 0 \text{ as } x \to \infty \\ x^2\dfrac{dY}{dx} + zb, \qquad x^2\dfrac{dy}{dx} + b \to 0 \text{ as } x \to 0 \end{array}\right\} \quad . \quad . \quad (3.22)$$

The function $f(Y)$ can be expanded in powers of $Y^{\frac{1}{2}}$, and its first two terms are $1 + Y$. If, then, we neglect $Y^{\frac{3}{2}}$ and higher powers on the right-hand side of (3.20), we obtain the Debye-Hückel equation, with the solution

$$Y \sim \frac{zb}{x}e^{-x} \quad . \quad . \quad . \quad . \quad (3.23)$$

Similar treatment of (3.21) then gives

$$y \sim \frac{b}{x}e^{-x} \quad . \quad . \quad . \quad . \quad (3.24)$$

This can be seen by introducing

$$\eta = zy - Y \quad . \quad . \quad . \quad . \quad (3.25)$$

so that

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{d\eta}{dx}\right) = -\frac{1}{z+1}\{ze^{-y} - e^{-z\mathrm{Y}} - (z-1)f(\mathrm{Y})\}\,. \quad (3.26)$$

with

$$\eta \to 0 \text{ as } x \to \infty \quad \text{and} \quad x^2\frac{d\eta}{dx} \to 0 \text{ as } x \to 0\,. \quad (3.27)$$

The terms retained in the bracket amount to $-\eta$, and the solution is

$$\eta \sim 0\,. \quad (3.28)$$

For the second approximation, we take over to the left-hand side of (3.20), (3.26) the terms previously retained.

$$\frac{d}{dx}\left(x^2\frac{d\mathrm{Y}}{dx}\right) - x^2\mathrm{Y} = -x^2\Phi(\mathrm{Y}) \quad (3.30)$$

$$\frac{d}{dx}\left(x^2\frac{d\eta}{dx}\right) - \frac{x^2\eta}{z+1} = -\frac{x^2\phi(y,\mathrm{Y})}{z+1} \quad (3.31)$$

where

$$\Phi(\mathrm{Y}) = \frac{1}{z+1}\{e^{-z\mathrm{Y}} - 1 + z\mathrm{Y} - f(\mathrm{Y}) + 1 + \mathrm{Y}\} \quad (3.32)$$

$$\phi(y,\mathrm{Y}) = z(e^{-y} - 1 + y) - (e^{-z\mathrm{Y}} - 1 + z\mathrm{Y}) - (z-1)(f(\mathrm{Y}) - 1 - \mathrm{Y}) \quad (3.33)$$

In Φ, ϕ, which were previously neglected, we replace Y, y by their approximate values (3.23), (3.28), and solve (3.30), (3.31) by means of the Green's functions of the left-hand side. Thus by (3.30), (3.22),

$$\mathrm{Y} = \frac{zb}{x}e^{-x} + \frac{1}{2x}\int_0^\infty \{e^{-|t-x|} - e^{-t-x}\}\Phi\left(\frac{zb}{t}e^{-t}\right)tdt \quad (3.34)$$

We require the potential due to charges other than the central ion.

$$\mathrm{Y}' = \lim_{x\to 0}\left(\mathrm{Y} - \frac{zb}{x}\right) = -zb + \int_0^\infty e^{-t}\Phi\left(\frac{zb}{t}e^{-t}\right)tdt \quad (3.35)$$

Similarly if we write

$$\lambda^2 = z + 1, \qquad x = \lambda\xi \quad (3.36)$$

(3.31) reduces to the same form as (3.30); and, using (3.27),

$$\eta' = \lim_{x\to 0}\eta = \int_0^\infty e^{-t}\phi\left(\frac{b}{\lambda t}e^{-\lambda t}, \frac{zb}{\lambda t}e^{-\lambda t}\right)tdt \quad (3.37)$$

An attempt was made to approximate to these integrals on the assumption that zb was small. This would be appropriate to some stellar atmospheres. For example, at the centre of a giant of type F and mass 10 we have roughly * $\mathrm{T} = 10^7$, $\mathrm{N/V} = 10^{20}$, $z = 23$; so that $\kappa = 10^7$ and $b = 2.10^{-3}$.

The lower limit of the integral in (3.35) is replaced temporarily by an infinitesimal ϵ; and Φ is split into three parts, consisting of the

* See Eddington, *loc. cit.*, p. 136; Fowler and Guggenheim, *M.N.R.A.S.*, **85**, 945.

first two terms in its expansion and the remainder of the series. From (3.32)

$$(z+1)\Phi(Y) = Y^{\frac{3}{2}}/\Gamma(\tfrac{5}{2}) + (z^2-1)Y^2/2 + (z+1)\Phi'(Y)\,. \quad (3.40)$$

The contributions of the three parts to (3.35) are (omitting infinitesimals)

$$\frac{1}{z+1}\int_\epsilon^\infty \frac{(zb)^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}e^{-\frac{1}{2}t}t^{-\frac{1}{2}}dt = \frac{4}{3}\sqrt{\frac{2}{5}}\frac{(zb)^{\frac{3}{2}}}{z+1} \quad . \quad . \quad . \quad (3.41)$$

$$\frac{1}{z+1}\int_\epsilon^\infty \frac{z^2-1}{2}(zb)^2e^{-3t}t^{-1}dt = \frac{1}{2}\frac{z^2-1}{z+1}(zb)^2(\log 1/3\epsilon - C) \quad (3.42)$$

where C is Euler's constant 0·5772 . . .

$$\int_\epsilon^\infty e^{-t}\Phi'\Big(\frac{zb}{t}e^{-t}\Big)tdt = \int_0^{zb/\epsilon}\Phi'(Y)(zb)^2\frac{e^{-3t}}{1+t}\frac{dY}{Y^3} \quad . \quad (3.43)$$

In the last integral t is put equal to 0. The argument for this is that t is small over most of the range of Y. zb being small, we can choose τ so that $zb<<\tau<<1$. Then for $Y>zb/\tau$, $t<\tau$ and may be neglected. The total error in the integral between $Y = 0$ and $Y = zb/\tau$ is less than

$$(zb)^2\int_0^{zb/\tau}\Phi'(Y)dY/Y^3 \quad . \quad . \quad . \quad (3.44)$$

Replacing $\Phi'(Y)$ by the first term in its expansion, which is

$$\frac{1}{z+1}\frac{Y^{\frac{3}{2}}}{\Gamma(\frac{7}{2})},$$

(3.44) appears to be small in comparison with $(zb)^2/z+1$. This argument, however, is fallacious unless z^2b is small; for owing to the exponent $-zY$ in (3.32), $\Phi'(Y)$ cannot properly be replaced by the first term in its expansion unless zY as well as Y is small. With $z = 23$ the difference between the two conditions is catastrophic.

To summarise the results of this approximation, omission of the awkward factor from (3.43) leaves

$$-\frac{1}{2}\frac{(zb)^2}{z+1}\left\{(z^2-1)\Big(\log\frac{zb}{\epsilon}+C-\frac{3}{2}\Big)+z^2\log z\right\} \quad . \quad (3.45)$$

Adding (3.41), (3.42), (3.45),

$$\Gamma' = -zb+\frac{4}{3}\sqrt{\frac{2}{5}}\frac{z^{\frac{3}{2}}b^{\frac{3}{2}}}{z+1}-\frac{1}{2}\frac{z^2b^2}{z+1}\{(z^2-1)(\log 3zb+2C-\tfrac{7}{2})+z^2\log z\} \quad . \quad (3.46)$$

A similar approximation to (3.37) gives

$$\Phi' = \frac{z^{\frac{3}{2}}b^{\frac{3}{2}}}{z+1}\cdot\frac{4}{3}\frac{z-1}{\Big(\frac{3}{2}+\frac{1}{\lambda}\Big)^{\frac{3}{2}}}$$

$$+\frac{1}{2}\frac{zb^2}{z+1}\{(z^2-1)(z+1)(\log(2+1/\lambda)b+2C-\tfrac{3}{2})+z(2z^2+z-1)\log z\} \quad (3.47)$$

In finding the electrostatic term in the work function, A, each charge is reduced by a factor t. This is accomplished by replacing ϵ by ϵt in all our formulæ. Alternatively, we may consider ϵ as a variable, when (2.21) shows that

$$\frac{\partial A}{\partial \epsilon} = Nz\Psi' - zN\psi' \quad . \quad . \quad . \quad (3.50)$$

or

$$\frac{\epsilon}{zNkT}\frac{\partial A}{\partial \epsilon} = Y' + y' = \frac{(z+1)Y' + \eta'}{z} \quad . \quad . \quad (3.51)$$

Now ϵ enters into Y' and η' only through b, which $\propto \epsilon^3$. Therefore

$$\epsilon\frac{\partial A}{\partial \epsilon} = 3b\frac{\partial A}{\partial b} \quad . \quad . \quad . \quad . \quad (3.52)$$

which with (3.51) gives $A - A_0/zNkT$ in terms of b and z, where A_0 is the work function for zero charges. The pressure is then given by

$$p = -\frac{\partial A}{\partial V} \text{ at constant temperature} \quad . \quad . \quad (3.53)$$

Differentiation of A_0 gives p_0, the uncorrected pressure; and V enters into $A - A_0$ only through b, which $\propto V^{-\frac{1}{3}}$. Therefore

$$(p - p_0)V = -V\frac{\partial(A - A_0)}{\partial V} = \frac{1}{2}b\frac{\partial A}{\partial b} = \frac{1}{6}\epsilon\frac{\partial A}{\partial \epsilon} \quad . \quad (3.54)$$

by (3.52). Using (3.51), (3.46), (3.47),

$$\frac{(p - p_0)V}{zNkT} = \frac{(z+1)Y' + \eta'}{6z} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.55)$$

$$= -\frac{z+1}{6}b + \frac{z^{\frac{3}{2}}b^{\frac{3}{2}}}{z+1}\frac{2}{9}\left[\frac{1 - \frac{1}{z}}{\left(\frac{3}{2} + \frac{1}{\lambda}\right)^{\frac{1}{2}}} + \frac{1 + \frac{1}{z}}{\left(\frac{5}{2}\right)^{\frac{1}{2}}}\right]$$

$$- \frac{1}{12}\frac{z^2 - 1}{z+1}b^2\left\{(z^2 - 1)(\log b + 2C - \tfrac{3}{2}) + z^2\log z^2\right.$$

$$\left. - \left(\frac{z^2}{z} + 1\right)\log\left(2 + \frac{1}{\lambda}\right) + \left(z^2 + \frac{z^2}{z}\right)\log 3\right\} . \quad . \quad (3.56)$$

The left-hand side is the ratio of the correction to the pressure to the partial pressure of the electrons. The first term on the right is that given by the simple theory. The condition $z^2b << 1$ renders the result useless for large values of z such as are found in the stars. Thus with $z = 23$, b must not be greater than about 10^{-4}. With these values the first term on the right of (3.56) is -4.10^{-4}, and the remaining terms give 2.10^{-5}.

The form of (3.56) is suitable for modification for the general case in which there are many kinds of ion. zN must be replaced by $\Sigma_i z_i N_i$,

and on the right-hand side z, $z^{\frac{3}{2}}$, z^2, $z^2 \log z^2$ are replaced wherever they occur by their mean values. In calculating these means, the ι^{th} kind of ion is given a weight proportional to $z_\iota N_\iota$. Similarly

$$\lambda^2 = \bar{z} + 1 \quad . \quad . \quad . \quad . \quad (3.57)$$

$$\kappa^2 = \frac{4\pi\epsilon^2}{VkT}(\Sigma_\iota z_\iota^2 N_\iota + \Sigma_\iota z_\iota N_\iota) = \frac{4\pi\epsilon^2}{kT}\frac{\Sigma_\iota z_\iota N_\iota}{V}(\bar{z} + 1) \quad . \quad (3.58)$$

Particles of High Velocity in the Chromosphere.

By R. W. Gurney, Ph.D.

(Communicated by Dr. H. N. Russell.)

The absorption of a quantum of $\lambda 3969$ by an atom of ionised calcium in the chromosphere gives the atom a velocity of 2·5 cm. per second, of which the upward component is on the average 1·25 cm. per second. Now the value of gravity on the sun being $2{\cdot}7 \cdot 10^4$ cm. sec^{-2}, a body acquires a velocity of 1·25 cm. per second by falling for $4{\cdot}6 \cdot 10^{-5}$ seconds. So that to be held up by radiation-pressure the atom must, as Milne pointed out,* make $2{\cdot}2 \cdot 10^4$ absorptions (and re-emissions) per second. If the atom were to make only 2.10^3 absorptions per second it would fall with an acceleration 90 per cent. of that of gravity, that is to say almost unchecked.

The atoms given in the following table are, when in their normal quantum state, incapable of absorbing radiation of lower frequency than that shown in column 2. The relative number of quanta of each of these frequencies contained in the solar continuous spectrum, calculated from Planck's formula, are given in column 3, taking the intensity at $\lambda 3969$ as unity.

Ionised Calcium	λ 3969Å	1
Neutral Helium	584Å	10^{-13}
Double ionised Ca	< 600Å	2.10^{-13}
Ionised Sodium	< 400Å	3.10^{-21}
Ionised Hydrogen	none	0
Neutral Hydrogen	1216Å	10^{-5}

The intensities of these frequencies are so small that their radiation-pressure could not support atoms unless the absorption coefficients of the latter were incredibly greater than that of ionised calcium for $\lambda 3969$. Thus, when a helium atom returns to its lowest quantum state, when calcium becomes doubly ionised, when hydrogen or sodium becomes ionised—in each case the atom will fall under gravity until its motion is reversed by collisions with other atoms.

The density in the upper chromosphere must be extremely low. It

* *M.N.R.A.S.*, **84**, 354.

appears from Milne's equilibrium calculations * that at a height of 9000 kilometres the density may be about 3.10^{-18} gr. cm.$^{-3}$. At this density the molecular mean free path would be more than 50,000 kilometres. Now suppose that a calcium ion, having become doubly ionised at a height of 9000 kilometres, falls to a height of 3000 kilometres. The time taken to fall this distance under gravity would be 210 seconds. We proceed to calculate the velocity acquired.

$$\begin{aligned} V^2 &= 2 \,.\, g \,.\, s \\ &= 2 \,.\, 2{\cdot}7 \,.\, 10^4 \,.\, 6.10^8 \\ &= 32.10^{12} \end{aligned}$$

The velocity acquired is therefore nearly 60 kilometres per second, in contrast to the 1·7 kilometres per second of the ordinary temperature velocity. The energy acquired is 640 volts.

In the absence of gravity we should have equilibrium between Ca^{++}, electrons, and Ca^{+}, according to the Saha theory; in any second a number of atoms would become doubly ionised, and this would be compensated by the recombination of an equal number of Ca^{++} ions with electrons. But in the presence of gravity the process of ionisation is uncompensated, and the Ca^{++} ions are continually removed by gravity as soon as formed.† The neutralisation of these ions upon their arrival in the lower layers will produce a "sink" of electrons there. A downward diffusion of electrons will ensue, accompanied by a small electrostatic field. Milne has pictured the equilibrium of the chromosphere as an almost static balancing of gravity against radiation-pressure. Rather we seem to require a slow general upward movement under excess radiation-pressure to compensate the falling of atoms which are continually getting into quantum or ionised states wherein they are incapable of being supported.

It appears therefore that the lower chromosphere will be subjected to a continual rain of falling atoms of calcium, helium, and hydrogen, falling both from the upper layers and from the numerous small prominences with their streamers. Atoms falling from the latter will penetrate into the chromosphere. The energy acquired in falling only 20,000 kilometres will be more than 2000 volts for calcium, 200 for helium, and 50 for hydrogen.

The impacts which the falling atoms make may be either elastic or inelastic, resulting in the excitation of the atom. Hydrogen and helium atoms which collide with heavy atoms may be reversed in direction without appreciable loss of energy, and flung up again to a height near that from which they fell. Calcium ions colliding with hydrogen or helium atoms can give the latter a velocity double their own. These atoms falling and rebounding with a velocity of only 40 km. per second

* *M.N.R.A.S.*, **85**, 126.

† The whole continuous spectrum beyond $\lambda 1040$ is available for removing a second electron from Ca^{+}. This should be contrasted with the calculated paucity of monochromatic radiation which alone can excite a single atom—a fraction of an Ångstrom presumably much narrower than a whole observed absorption line. The radiation available for ionisation is comparatively plentiful.

can give by their Doppler effect, a line more than 1 Å wide in the visible region.

Of greater interest is the possibility of excitation by inelastic collisions. We know that the collision of an excited atom with another molecule may result in the conversion of the energy of excitation into the kinetic energy of the colliding atoms. This is known from the quenching of mercury fluorescence by the admixture of argon, etc.* And the "principle of detailed balancing" seems to require that the converse process must be capable of occurring—the conversion of the kinetic energy of colliding atoms into the excitation of one of them. This process would be very welcome in the chromosphere if, for example, it would raise hydrogen atoms to the Balmer states, and so increase the concentration of atoms capable of absorbing the Balmer lines.

In the more violent impacts we may have a mechanism for the production of excited and ionised helium, more efficient than that by absorption of radiation. For suppose that the absorption coefficient of helium for $\lambda 584$ is of the same general order of magnitude as that of Ca^+ for $\lambda 3969$, we have seen that the intensity of solar radiation in the visible region is sufficient to cause Ca^+ to absorb at intervals of 10^{-5} of a second. Since the quanta of $\lambda 584$ are 10^{13} less plentiful, the interval between successive absorptions would be $10^{-5} \times 10^{13} = 10^8$ seconds for helium in its lowest quantum state. That is to say, it would have to wait about three years, unless in the meantime it were excited by collision or some other agency.

The continued existence of prominences and of the upper chromosphere are matters of observation; and the continual shower of atoms and ions falling from them seems to be a necessary consequence. The energy required to lift the material is probably derived from the outflowing stream of radiation; and the fall under gravity provides a mechanism whereby the energy accumulated by a number of absorptions of comparatively long wave-length may be delivered at a single blow. The discussion applies generally to all stars at a similar temperature which possess chromospheres.

It may be pointed out here that very great energies will be acquired by particles, meteoric or otherwise, which fall to the surface of a "white dwarf." With the accepted values of the density and radius of the Companion of Sirius (30,000 and 0·03 times that of the sun respectively), we find that an atom of iron falling to the surface would acquire an energy of more than three million volts. The energy of a similar particle falling from infinity into the sun would be 100,000 volts.

In conclusion, I wish to express my sincerest thanks to Dr. Russell for his kind interest and help, and to the International Education Board to whose grant of a fellowship my visit to America was due.

* Stuart, *Zeit f. Physik*, **32**, 266.

Princeton University.

Absorption by Nebulosity and the Temperature and Luminosity Sequences of Novæ. By M. C. Johnson.

1. *Introduction.*—The problem of constructing a theory of Novæ to fit the observed facts can be identified nowadays with the problem of understanding why the photosphere of a low-luminosity star should rapidly brighten and then be dimmed again. Milne has shown (1) that, once given this temporary brightening, Doppler effects in an absorption spectrum such as are common to all Novæ are precisely what would be expected in the light of recent advances in the theory of line absorption.

Theories to account for this brightened photosphere must be criticised on two main grounds: (*a*) the adequacy of a supposed cause to liberate sufficient energy temporarily; (*b*) the probability of the supposed cause arising often enough in the stellar system to account for the frequent occurrence of Novæ.

We might divide possible theories according as they place the immediate cause of brightening outside or inside the star. In the last few years Hartmann and Lunt (2) have revived the interior cause hypothesis, originally due to Evershed. But in the present state of atomic physics, any theory of an outbreak due to sudden increase in the emission of energy from a star's interior leaves us completely without ground for discussion by either of the criticisms we have called (*a*) or (*b*).

Accordingly the present paper is an attempt to open for consideration certain possibilities of the exterior cause hypothesis which have not so far been explored. It is hoped that they may prove more capable of being compared with known facts both as regards (*a*) and (*b*) than theories which have to depend on our ignorance of the conditions governing liberation of subatomic energy.

2. *Matter available for Collision.*—Hypotheses of an external cause have laid themselves open most readily to criticism (*b*). Thus, there is no doubt that a collision between two stars would generate enough surface heat to satisfy (*a*), but from what we know of stellar velocities and free paths, the rarity of such an occurrence prohibits our considering it seriously as the origin of most Novæ. Hence the far larger target area provided by extended cloud of any kind limits possible collision theories to nebulosity or meteor showers. To satisfy the criticism of type (*a*) attention has so far been directed to the alternative of a meteor shower. Halm (3) has shown that the heat developed when a dark star meets a shower of meteors would equal that normally radiated by the sun, if we assume a suitable density for the shower. The photosphere would be brightened suddenly, and we could understand the subsequent events by the method indicated by Milne; the Nova sequence would be complete in outline.

But criticism of type (*b*) has been neglected. Is there any evidence for the existence of meteor showers outside the solar system? Eddington has more than once commented (4) on the difficulty of postulating

anything but gaseous material in interstellar space. The only matter which seems at present capable of satisfying criticism (*b*) as a likely target for wandering dark stars is gaseous nebulosity. We proceed therefore to explore, with due regard for (*a*), a hypothesis that the motion of a faint star in or into gaseous nebulosity could give rise to a local liberation of energy resulting in a temporarily brightened photosphere.

According to Eddington's work on interstellar matter, most of space is filled with nebulosity of a density of the order of 10^{-24} gm. per c.c. This is at a high temperature (10,000°, say) because of the ability of electrons to replenish their kinetic energy in photo-electric ejection by general star-light, and the inability of widely separated atoms to dissipate the energy which they ultimately receive from those electrons. Diffuse nebulæ, whether bright (in the neighbourhood of the hotter stars) or dark, probably represent the beginning of condensations in the general cloud, of an order of density of 10^{-20} gm. per c.c. In turn Eddington postulates subcondensations of central density 10^{-17} gm. per c.c. Several lines of evidence suggest several estimates of size for these, of which the smallest is of the order of 0·14 parsec radius. Eddington supposes that before these condensations gather to centres as foci for the future formation of stars, their temperature will have fallen below that of the original nebulosity.

The interaction between a star of low luminosity and nebulæ of the above types will present features very different from the collision of star and meteor shower.

3. *The Meaning of Star-nebular "Collision."*—If by collision is meant the passage from a place outside the cloud, to contact with or entry into the cloud, it is at once apparent that collision is not a sudden phenomenon, even with the smallest subcondensation. Although Eddington warns us that treatment of these clouds as isothermal gas spheres is a rough approximation only, no refinement would provide them with anything approaching in properties a sharp edge with which to "collide," in the more familiar sense of the word. The sharp edge would only be present when a subcondensation had so far shrunk that it came under the ban of criticism (*b*) through lack of target area. A star of proper motion 20 km. per sec. meeting a subcondensation with similar motion in an opposite direction would take 3×10^3 years to reach the central regions, during which period the density of the medium would gradually rise perhaps a thousandfold. Accordingly no sudden discontinuity in the star's behaviour can be expected from the direct results of such a "collision."

4. *Accretion.*—The only primary interaction between the star and the nebulosity through which it makes its way is the accretion of material by the star's gravitational attraction.

Eddington (5) has demonstrated the inadequacy of the normal accretion process when regarded as a gain of mass balancing the loss by radiation in a luminous star moving through the universal interstellar cloud of density 10^{-24} gm. per c.c. According to his equation,

$$dM/dt = 2\pi G M \rho R/V \quad . \quad . \quad . \quad . \quad (1)$$

where G is the gravitational constant, ρ is the nebular density, M and R are mass and radius of the star, and V is its velocity relative to the nebulosity. Hence a star of solar mass and size and velocity moving in the central regions of a subcondensation would gain material at the rate of 3×10^{14} gm. per sec. The solar loss of mass by radiation is 4×10^{12} gm. per sec. It is easy to show that for a star of effective temperature 3000°, such as may represent the pre-Nova state, loss by radiation is negligible compared with accretion in passing through a medium of density 10^{-17} gm. per c.c., such as a subcondensation ; the loss would balance accretion in this case at 8×10^{-21} gm. per c.c.

Brown's theory (6) of the geometrical configuration of the regions of a nebula attracted by a star familiarises us with the possibility of a relative velocity as low as 2 km. per sec. The accretion for a solar star in the above dense medium at this velocity would be 3×10^{15} gm. per sec.

Eddington points out that at relative velocities of star to nebula less than the velocity of sound (4 km. per sec.) accretion will probably rise much faster than expected on the above simple theory, owing to the advance of a compression wave in front of the star. Accretion will be still further increased when the capture target of the star is measured by the radius of its outermost appendages. These, in a dark star, will be inappreciable until a considerable time of accretion has already passed. In view of these variable additions to the normal accretion rate, it is not unlikely that a star of solar mass and size, but lower luminosity, moving at 2 km. per sec. in a medium of density 10^{-17} gm. per c.c., would gain 10^{16} gm. per sec. This will be used as a maximum estimate.

The mass of all the extra-photospheric layers of the sun is about 0·4 gm. per cm.^{2}. The solar radius is 7×10^{10} cm. and its surface area is 6×10^{22} cm.^{2}. At the given maximum accretion rate, matter equivalent to the whole of its layers which absorb or emit line spectra would be added in 28 days. This must not, however, lead us to regard even such accretion as seriously altering the mass type of the star we are investigating. The solar mass of 2×10^{33} gm. would take nearly 10^{10} years to double itself by this process, by which time the star might have emerged from the denser regions, or the nebulosity itself might be condensing into stellar proportions.

5. *Energy becoming available during Accretion.*—Not only is collision between star and nebulosity a slow process, but the energy transferable during accretion is very small compared with the energy given to the star by the meteor shower in Halm's theory. The kinetic energy available from the transfer of matter from nebula to star is, at its greatest, equivalent to the stopping of the maximum 10^{16} gm. gained in one second, from a velocity equal to the relative velocity of star to nebula. At 2 km. per sec. this is 2×10^{26} ergs. But the energy radiated by the sun is $3{\cdot}8 \times 10^{33}$ ergs in the same time. A similar star, near the lower limit of visibility, say at 3000°, would still radiate more than 2×10^{32} ergs per sec., or more than a million times the energy available from the accretion.

Eddington regards diffuse nebulosity as a great store of potential energy, for the above-mentioned reason of the properties of photo-electrons. But unless the dark star in the subcondensation could in some way act as a focus, draining this accumulated energy from a very large region of surrounding cloud, this is an even less effective agent for heating the former's photosphere, than the retarding of captured matter. At the maximum density of 10^{-17} gm. per c.c. even high temperature matter could not transfer its energy rapidly without incredibly steep radiation intensity gradients, since conduction by collisions at such density would be negligible.

We further can neglect energy liberated by the retardation of the star itself, since not only the density of the medium but its density gradient is very small.

6. *Absorption in an Accretion Layer.*—Having rejected the energy of the medium as a source of energy for a Nova outbreak, we turn instead to the normal radiation from the star itself. The intrusion of the medium into a star's regular life will result in repression of the quite considerable flux of energy which should emerge from even a star of low luminosity. The postponed liberation of this normal energy after a long period of imprisonment may, as a secondary result of accretion, take place with a violence which we have shown is not to be expected in the primary results of the movement in the nebulosity.

We proceed to investigate a hypothesis that such liberation of the accumulated and suppressed normal radiation of a star is capable of giving rise to the output of energy seen in a Nova.

Two observational facts suggest that such cumulative, rather than primary, results of the passage of a star through nebulosity may be of interest in the theory of Novæ. Firstly, Brown's study of Hubble's nebula provides a visible case of a star moving through nebulosity with low velocity but without catastrophe. Secondly, it is usual for a Nova to die down to faintness again after its outbreak. The process leading to catastrophe is clearly neither a universal nor a lasting consequence of the interaction of star and cloud.

The normal flow of radiation from a star is determined by its mass and central temperature, and by the absorption coefficients of the layers through which the radiation is handed on before it reaches the medium surrounding the star. We have shown that (para. 4) the mass type must be considered approximately constant even during the greatest accretion, and we are not at present exploring any interior cause hypothesis of Novæ, *i.e.* we assume the state of the far interior is steady. The medium surrounding the star is in general of negligible obstructive power compared with even its outer atmosphere. The radiative equilibrium of any part of the star's material will determine a set of gradients of radiation intensity and of temperature proper to a definite set of absorption coefficients. If we were able to wrap round the outside of a star a dense absorbing blanket preventing free escape into the surrounding comparatively transparent medium, those gradients would be modified profoundly, as the blanket would itself reradiate to the photosphere. Such an absorbing blanket is being continuously

applied by the accretion layer in the case of a star moving in a nebulous condensation. The layers being added to the star by accretion will both dim the appearance of the star from outside and will themselves rise in temperature.

But if the energy which they prevent from entirely leaving the star accumulates, and ultimately is liberated in such a manner that we see the phenomena of a Nova, there must be a balance between what is liberated finally and what is cumulatively prevented from leaving the star during the accretion. Let t_1 be the time during which the star is emitting less than its normal radiation to outer space by reason of the obstruction of growing accretion layers, and let t_2 be the time during which it is emitting more than its normal radiation in a Nova outburst. Let E_1 be the mean rate of absorption of energy by the nebulosity during t_1, and let E_2 be the mean rate of emission of energy during t_2 over and above the normal emission rate of the star. Then if the criterion of theories of Novæ referred to in the introduction as of type (*a*) is to be satisfied by the present hypothesis, we must show that

$$E_1 t_1 = E_2 t_2 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or, the total decrease of emission before the outbreak must equal the total increase during the outbreak.

7. *Absorption Coefficients and the Mechanical Equilibrium of an Accretion Layer.*—The condensation or subcondensation through which we suppose a star to be moving is of a higher volume absorption coefficient than the Eddington nebulosity pervading all space, but the matter actually captured by the star's gravitational field may have far higher volume absorption coefficients still. It is the absorption of momentum from radiation which largely determines the mechanical equilibrium of the layers surrounding the star's photosphere. To investigate the quantity denoted above by E_1 we must determine the nature of the mechanical equilibrium taken up by the layers which add themselves to the star by accretion from the nebulosity through which it moves.

Milne has distinguished (7) between two types of stellar atmosphere : these will provide clues to the structure of the accretion layers. Let g be the acceleration of gravity at a star's surface, let F_λ be the intensity of radiation of wave-length λ crossing unit area in unit time, let c be the velocity of radiation and κ_λ a monochromatic mass absorption coefficient. Then the mechanical equilibrium of the atoms concerned with κ_λ is represented by the equation,

$$\frac{\kappa_\lambda F_\lambda}{c}/g = 1 - \mu \quad . \quad . \quad . \quad . \quad (3)$$

Then, if $\mu = 0$ we have matter behaving as a chromosphere or totally supported atmosphere. Equilibrium is a simple suspension of individual atoms between alternate absorptions and downward falls. There is no appreciable pressure gradient, and the usual hydrostatic pressure and density law is replaced by an almost inappreciable decrease to nebular orders of magnitude.

If, however, μ is small but distinct from zero, matter behaves as a reversing layer or partially supported atmosphere. There is a pressure gradient of the familiar isothermal type in which

$$p/p_0 = e^{-m\mu gx/RT} \quad . \quad . \quad . \quad . \quad (4)$$

Here m is an atomic mass, x is a distance along the star's radius, R is Boltzmann's constant, and μ and g have the same significance as before. The gravitational term in the exponential is reduced to a fraction of the true value of g by the upward acceleration of atoms absorbing momentum from radiation. This type of atmosphere will have, accordingly, a strongly condensed base in which the pressure and pressure gradient rise rapidly.

Now the values of κ_λ are known to be very high, of the order of 10^7 and 10^8 for all substances for which data are so far available; but such coefficients not only depend on the radiation being of a certain frequency, but require that the absorbing atoms shall be in certain states before they can absorb it. It has been shown by Milne (8) that such line absorption only determines the equilibrium of layers above the photosphere of a star. True photospheric and infra-photospheric equilibrium is determined in general by the smaller coefficients governing the absorption of continuous radiation by the mechanism of photo-electric ejection. Milne has shown (9) that for these Eddington's law

$$\kappa \propto T^{-\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

is not greatly in error down to solar effective temperature, but that in practice κ probably varies from the order of 10^2 to 10^3 in passing from the photosphere of a giant to that of a dwarf at the same temperature. For instance, he finds the value for κ for Ca at 6000° to be 1140 in the sun and 133 in the giant Capella.

There is abundance of atomic collisions as soon as we descend below the layers of line spectra in a star, and hence the opposition of gravitation and radiation-pressure is not as simple and complete in a photosphere as in the atmosphere for which μ is very small. In cases where we wish, however, to compare gravitational and radiative acceleration, neglecting the effect of gas-pressure in diffusing motions away from the radial direction, we consider equilibrium is possible so long as

$$\kappa\sigma T^4/c \leqq g \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Here the continuous absorption governed by κ requires the substitution of the total intensity of black-body radiation for F_λ, the monochromatic intensity; hence the available radiation is governed by T the photospheric temperature and σ Stefan's constant, equal to $5{\cdot}7 \times 10^{-5}$.

8. *The Case of a hot Star.*—Assuming on the basis of the preceding paragraph that the value of κ is 10^3 at 6000°, varies according to law 5 for higher temperatures, and is to be divided by the factor 8 for stars of definitely giant characteristics, it is possible to understand at once how an early type star could move through even a dense nebulosity without catastrophe.

Consider line absorption by its accretion layers. A star of type

A, B, or O will already possess an extended chromosphere of the kind in which μ is almost equal to zero. Milne's study (10) of Ca^+ shows that even at solar temperatures elements with the most favourably placed ionisation frequencies may at times be driven off the surface of a star. In a previous paper (11) I have shown that for the less favourably situated elements which are common in the spectra of the hottest stars, radiative acceleration may be far greater than gravitational acceleration.

Consider continuous absorption. The surface values of g in A-stars are known to range from $0{\cdot}2 \times 10^4$ to $2{\cdot}0 \times 10^4$, the extremes being associated with extreme giant and dwarf characteristics respectively. Opposing these accelerations by that due to radiation according to equation 6, we find that equilibrium is only possible if κ is equal to or less than the following critical values :—

T.	κ (dwarf).	κ (giant).
10,000°	1050	105
12,000°	530	53

On the assumptions at the head of this paragraph, we find that for a dwarf, κ should only have fallen to 710 when the temperature has risen from 6000° to 12,000°, and for a giant to about 90.

Accordingly we must not only expect a low-pressure chromosphere in stars of A or earlier type, but also a diffused photosphere in which radiation-pressure almost neutralises gravitation. In a stable star the apparent impossibility of any photospheric equilibrium at all at temperatures much above 10,000° will be subject to adjustment by the star itself. As in the chromosphere, there will be expansion resulting in a decrease of pressure and fall of the volume absorption coefficients.

Matter attracted from nebulosity outside the star will begin to interpenetrate these extended low-pressure regions, whose density, so far as the outer chromosphere is concerned, is of the same order as its own. But before any of the new matter reached the photosphere it would become subject to all the star's radiation, with the exception of certain absorption lines. Since at first the nebulosity may be expected to be at a low temperature, its absorption frequencies will not coincide with those of the stellar atmosphere, and it will receive the full impact of the radiation. It will for this reason absorb more, rather than less, than do the actual chromospheric gases, and it will act as an outer atmosphere fully supported by line absorption. Accordingly the new matter gained by accretion from the nebulosity will be held at a distance, and the density of the outer layers of the growing star will scarcely alter at all. The density will remain finite for a greater distance from the star than if the nebulosity were not there, and the radiation intensity gradient will with it remain finite, though small. If some of the frequencies for which this gradient persists happen to be in the visible region, we shall see the star surrounded by extensive bright nebulosity obeying the laws discovered by Hubble in his observations of diffuse nebulæ. There will not be any serious alteration of total radiation intensity gradients or density gradients in the matter which belonged to the

star before accretion. Hence the accretion will not result in a large accumulation of heat in the star itself; its own high temperature will prevent it from acting according to the hypothesis of para. 6, and we shall have no reason to expect a Nova outbreak.

9. *The Case of a cool Dwarf.*—The behaviour of the accretion layers from the same nebulosity, but for a dwarf at, say, 3000° effective temperature, will be very different from the above.

Consider the maximum line absorption possible. Elements which in the sun were reaching a stage of excitation enabling them to absorb visible radiation selectively, will here be transparent, except to the ultra-violet radiations appropriate to their normal quantum states. The ultra-violet intensity in the black-body spectrum of the 3000° photosphere will be extremely small, decreasing with decrease of temperature much faster than the total radiation, which will be one-sixteenth of the total radiation at 6000°. Hence the extension of gases, either of normal stellar origin or belonging to the accretion layer, to a chromospheric type of equilibrium, would not occur, even if we went to the extreme of ascribing hot star absorption coefficients to the nebulosity. The value of μ in equation 3 will be approximately unity for most elements. The outer gases of the star will be in an equilibrium scarcely distinguishable from that imposed by gravitational control of a gas-pressure gradient.

Turning to the region of continuous absorption we must at first take account of the maximum possible radiation-pressure. Suppose, therefore, that the law 5 held down to low temperatures (a supposition to be examined later), then at 3000° κ would be 1400, and if g were $2{\cdot}7 \times 10^4$ as in the sun, the acceleration due to radiation-pressure would be only 0·008 of that due to gravitation. Accordingly in the general equation of equilibrium in appreciable pressure gradients (4) μ would be again indistinguishable from unity.

Taking the example of an atom 40 times the mass of the hydrogen atom, assuming $T = 3000°$, $R = 1{\cdot}37 \times 10^{-16}$ and $g = 2{\cdot}7 \times 10^4$, we find from equation 4 that the factor by which pressure is multiplied in descending 10 km. towards the photosphere is 73. Since radiation-pressure is negligible this will hold for all approximately isothermal layers, *i.e.* all layers outside the immediate interior of the cool star, and for the accretion layers. This is a steeper gradient than in the luminous stars from which most of our observation of the properties of stellar atmospheres is derived, and in all of which radiation-pressure plays a large part. In particular the pressure gradient is steeper than in the sun. Milne has estimated (12) the thickness of the solar transition region between upper reversing layer and photosphere to be 31 km. with a pressure increase from 5×10^{-6} atmosphere to $1{\cdot}2 \times 10^{-4}$ atmosphere. Eddington (13) has made two estimates of photospheric thickness of 27 km. with a pressure increasing 16-fold and 17 km., on different assumptions. The thickness of the photosphere of the cool dwarf, defined as the layer from which, say, 90 per cent. of the star's continuous spectrum comes to us, will be less than that given by the smallest estimate for the solar photosphere.

We continue to use these studies of the normal structure of a star's outer layers as the clue to the equilibrium towards which the layers gained by accretion from nebulosity would tend. Accordingly, from the preceding considerations we conclude that the cool dwarf has no power to hold its accretion layers at a distance by line radiation-pressure as can the A, B, and O stars. There is nothing to prevent the new matter from falling upon the star and collapsing into the photosphere. When, as shown in para. 4, matter equivalent to the total extra-photospheric layers of the sun is added by accretion from the centre of a subcondensation to a cool star in 28 days, this matter will adopt not a reversing layer nor a chromospheric behaviour, but that of a photosphere with very steep pressure gradients.

10. *The Equivalence of Energy absorbed in Accretion and emitted in Nova Outbreak.*—We can now use the data and conclusion of the preceding three paragraphs in attacking the problem of para. 6. That is to say, we confine our attention to the cool dwarf, since we have seen in para. 8 that a hot star will not accumulate the dense accretion layer that characterises the star of 3000° effective temperature.

Consider first the right-hand side of equation 2, *i.e.* the energy emitted by a Nova during its outbreak. Now the assignment of temperatures to a Nova at its different stages is not free from ambiguity. We see successively in most Novæ line spectra which are associated in stable stars with A and B types, *i.e.* with temperatures of 10,000° and 15,000° in turn. But since the Doppler effects in Novæ indicate a great expansion of the outer gases, the change from A to B spectrum, generally occurring after maximum luminosity, may to a considerable extent represent the attainment of higher excitation stages consequent on the attainment of lower pressure, and may not be due to further rise in temperature. The duration of the very bright stage of a Nova lasts generally a few weeks only. To make an estimate of the quantities in equation 2, we shall construct a hypothetical average Nova emission. The difference between this mean and the emission for any ordinary Nova should not be greater than the differences shown by Novæ among themselves. We will take this mean Nova as a body emitting energy corresponding to a black body at 11,000° for a period of a quarter of a year, or $0{\cdot}8 \times 10^7$ sec. We assume the pre-Nova condition is that of a 3000° dwarf. The normal rate of emission is in this case negligible compared with E_2, the rate of emission of the additional energy.

The sun emits $3{\cdot}8 \times 10^{33}$ ergs per sec. Hence the total emission of the mean Nova for the duration of its increased luminosity will be this amount, multiplied by the ratio of the fourth power of the temperatures, and by the time.

$$\begin{aligned} E_2 t_2 &= 3{\cdot}8 \times 10^{33} \times (11{,}000^4/6000^4) \times 0{\cdot}8 \times 10^7 \\ &= 3{\cdot}4 \times 10^{41} \text{ ergs.} \end{aligned}$$

This quantity accordingly represents the right-hand side of equation 2.

The first requirement in considering the left-hand side of equation 2, *i.e.* the energy prevented by the accretion layers from leaving the star, is some assumption as to an absorption coefficient. But here we are

met by a difficulty, for the law 5 cannot be expected to be valid down to indefinitely low temperatures. It is known to describe mainly the mechanism of photo-electric absorption, and the effectiveness of such a mechanism implies a considerable partial pressure of free electrons. But at temperatures such as 3000° dissociation of compounds is still incomplete and ionisation rare. Hence our complete ignorance of molecular absorption coefficients makes it impossible to do otherwise than start from a speculative mass absorption coefficient, in order to put forward a numerical case for discussion. Let us assume that over the period of accretion before the Nova outbreak the mean mass absorption coefficient of the accretion layers is 50. In the early stages it may be much lower and in the final stage it may rise to the order of 10^3, known to be usual in a dwarf's photosphere at 6000°.

A second quantitative starting-point is the limit to which the star's luminosity might have fallen before it begins to rise in the Nova outbreak. If our hypothesis is true, there must be a dimming before outbreak, in which the total energy prevented from leaving the star is equal to the extra energy emitted during the subsequent brightening, according to equation 2. But Novæ are generally extremely faint before outbreak, of the order of 10 in magnitude; in no case is accurate measurement of previous luminosity over a long period of time available, except in the case of Nova Pictoris, where there is some unconfirmed evidence of a dimming before outbreak (14). It is not likely, therefore, that a 40 per cent. decrease of normal luminosity gradually accumulating over many years could be observed, superposed on the intrinsic faintness of the star, though the growth of variable star data may allow this test in the future.

We will take the case of a 3000° star whose normal energy emission E_0 has fallen to 60 per cent. of its original strength before brightening begins. From the normal solar radiation we have,

$$\begin{aligned} E_0 &= 3{\cdot}8 \times 10^{33} \text{ ergs per sec.} \times (3000^4/6000^4) \\ &= 2{\cdot}4 \times 10^{32} \text{ ergs per sec.} \end{aligned}$$

Let E be the radiation emerging from the outer surface of the accretion layer just before the Nova outbreak begins. Then

$$\begin{aligned} E/E_0 &= 0{\cdot}6 \\ &= e^{-\kappa\rho x} \\ &= e^{-\kappa m} \end{aligned}$$

where ρ is density and x is length of absorbing column of accretion layer, κ is mass absorption coefficient, and m is the mass per cm.^{2} of the column of accretion layer. For $\kappa = 50$ we find $m = 10^{-2}$ gm. per cm.^{2}.

Now if the rate of growth of absorption had been constant, which we assume as a first approximation, we can find the mean rate of absorption during accretion, E_1 of equation 2.

$$\begin{aligned} E_1 &= (E_0 - E)/2 \\ &= 0{\cdot}48 \times 10^{32} \text{ ergs per sec.} \end{aligned}$$

Then, since $$E_1 t_1 = E_2 t_2$$
$$= 3{\cdot}4 \times 10^{41} \text{ ergs.}$$
$$\therefore \quad t_1 = 7{\cdot}1 \times 10^9 \text{ sec.}$$
$$\fallingdotseq 200 \text{ years.}$$

Hence if the energy required for a Nova outbreak is estimated as equivalent to emission at 11,000° for a quarter of a year, it can be accumulated by 200 years of accretion of matter whose absorption coefficient is 50, the maximum dimming of the 3000° photosphere before outbreak being 40 per cent.

Since the final mass per unit area of the absorbing layer is 10^{-2} gm. per cm.^{2}, spread over an area equal to the sun's surface, 6×10^{22} cm.^{2}, we have a total accretion mass of 6×10^{20} gm. Since $t_1 = 7{\cdot}1 \times 10^9$ sec. this requires accretion at the rate of $0{\cdot}85 \times 10^{11}$ gm. per sec. This is far below the maximum possible accretion rates expected in the densest Eddington subcondensation, used previously in investigating the maximum possible "collision" energy of star and nebula. If we assumed the loss of mass by radiation was negligible, the calculation of para. 4 shows that for a solar mass and size and the slow relative velocity there assumed, the nebular density for the above accretion would be $2{\cdot}8 \times 10^{-22}$ gm. per c.c. Actually, as shown before, a nebular density of nearly 30 times this would be necessary before accretion overcame loss of mass by radiation. Hence the dark star would have to have moved well into the central regions of an ordinary dark nebula (10^{-20} gm. per c.c.) before loss by radiation was first neutralised, and after that the above processes, leading ultimately to a Nova outbreak, might be expected to begin.

11. *Differences between Luminosity and Temperature Sequences.*—Any theory of Novæ must recognise the great expansion of the outer layers measured by the Doppler effect. It is to be expected that not only the gas showing line spectra but also the gas showing a continuous spectrum should be spreading out and enlarging the star's luminous disc. For if radiation-pressure accounts for the expansion of the former according to Milne's theory, the region of the continuous spectrum is probably also a temporary seat of radiative accelerations greater than those due to gravity. When we compare the observed luminosity sequences of Novæ with a theoretical sequence derived solely from a temperature change, we do find a discrepancy; this discrepancy would be removed if we postulated an increase in total luminosity, due to such an increase in the surface area presented by the layers taking part in radiative processes.

In choosing 11,000° as a hypothetical mean effective temperature averaged over the quarter of a year which covers the outbreak, we allow for a peak on the temperature-time curve whose height could only be calculated if the shape of the curve were definite. A maximum luminosity stage lasting only a day or two suggests that a peak temperature of, say, 16,000° would still satisfy the total energy values we have assumed for the whole time of the outbreak. The change in luminosity of the star from minimum to maximum (neglecting the preliminary

dimming) would then be one of $16^4/3^4$, or approximately 800, or a change of between seven and eight magnitudes. Now, Novæ of not greater than 10th magnitude before outbreak sometimes reach magnitude 0 or an even brighter stage at maximum. This leaves a luminosity factor of about 16 not accounted for by the given rise in temperature of a body of constant surface area. A fourfold expansion in the disc of gases from which we have light would account for this further increase in luminosity. Such expansion is not by any means improbable in view of the fact that the observed velocity of the outrushing gases is as high as 1800 km. per sec. If the initial radius were that of the sun, 7×10^{10} cm., at a rate of 10^3 km. per sec. the gases could spread to four times that distance from the centre in less than an hour.

12. *Differences between a Star in Process of Accretion and a Star in a steady State.*—We have shown that criterion (*a*) of para. 1 is satisfied by the hypothesis that a Nova could originate in the passage of a dark star through gaseous nebulosity. That is to say, the energy stored up is sufficient to provide the energy estimated to be emitted during the Nova outbreak, and the observed temperature and luminosity sequences are not outside the range of probable consequences to be expected from a reasonable estimate of the accretion.

Since a low temperature and a slow motion relative to any dark nebulosity of not unusual density is sufficient to bring about the required density of accretion layer, we can also consider we are not violating criterion (*b*).

The next stage in a complete theory of Novæ would be to inquire why the outbreak is short and intense while the accumulation was slow and gradual, and what physical condition initiated the transition from one to the other. Neither of these questions seems susceptible of quantitative treatment at present: the present discussion is concerned solely with raising the hypothesis into the region of consideration by showing that it can satisfy the primary criteria of (*a*) and (*b*). But both further inquiries would be seen to be answerable in general terms if there were reason to expect the rate of absorption in some of the participating layers to be unstable. That is to say, that an increase in absorption led to further increase, *i.e.* to an acceleration of the heating up of certain layers.

Now, in the calculation of para. 10 we assumed speculatively a mean mass absorption coefficient of 50. During the earlier stages of accretion the absorption is probably less and in the later stages more. How much less is not calculable because of our ignorance of the coefficients concerned in molecular absorption at low temperatures. But we can say qualitatively that as the accretion layer grows, an absorbing and reradiating mass takes the place of the comparatively transparent space which previously received most of the star's radiation. Hence the photosphere rises in temperature and adopts some of the behaviour previously characterising lower layers. At some stage in the growth of the accretion layer the photosphere reaches a temperature at which photo-electric ionisation becomes the predominant mechanism in its opacity. Its rate of absorption then, so far as it depends on atomic

states and intensity of radiation, reaches the value appropriate to that temperature in a stable star. For instance, the mass absorption coefficient rises to the order of 10^3. But unless free expansion were at once possible to the lower pressure also appropriate to a layer at that temperature in the stable star, the volume absorption coefficient must rise further and faster, and with it the temperature, since the flux from below is not diminished. Both volume absorption coefficient and temperature may rapidly greatly exceed the values allowable in stability. Free expansion is, however, delayed by the presence above of a growing accretion mass, itself of a lower absorption coefficient and at the comparatively high pressure appropriate to the previous purely gravitational equilibrium. Increases in photospheric absorption coefficient and in photospheric temperature accordingly temporarily reinforce each other, a rise in the one quantity producing a rise in the other, until a breach in the superposed cool accretion layers is needed, to reduce the volume absorption coefficient of the photosphere to a value suited to equilibrium.

13. *Summary.*—In considering theories of Novæ involving an exterior cause, Eddington's types of nebulosity must nowadays be regarded as more probable targets for collision with a star than meteor showers. But the density gradients in such nebulosity are so slight that "collision" can only be a very slow process. Also the maximum energy given to a star by the densest nebulosity through which it moves is very small compared with even a dark star's normal radiation. But the star will acquire a new covering by accretion of matter from the nebulosity at a given rate.

We investigate the absorption of the star's normal radiation by these accretion layers.

In the case of a hot star the new layers will be held at a distance by radiation-pressure; the gradients of radiation intensity, temperature, and density in the star's appendages will not alter appreciably, and no Nova outbreak is to be expected. In the case of a cool dwarf, the accretion layers fall into the photosphere, and there will result a rise in temperature and an accelerated and unstable rise in absorption coefficient in the photosphere. This leads, under certain conditions, after a long period of accretion, to a breach in the accretion layer, allowing the postponed reduction of the absorption coefficient to a normal value. Balancing the energy absorbed during the long period of accretion with the energy estimated to be emitted during the short period of great luminosity, we find that the former would proceed for 200 years to account for an average Nova outbreak, in nebulosity of the accepted density of dark nebulæ. The long and gradual dimming before the brightening, which balances the luminosity during the short outbreak, is scarcely to be detected in a star intrinsically faint.

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Liquid Stars—A Correction. By J. H. Jeans.

Vogt * has pointed out an error in my equations of motion of stellar interiors which seems, *prima facie*, to invalidate my proof that gaseous stars are unstable. But I believe that the proof can easily be modified so as to lead to my original result.

As Vogt points out, the true equations of motion are

$$\frac{d^2r}{dt^2} = -\frac{1}{\rho}\frac{d}{dr}(p_G + \tfrac{1}{3}aT^4) - \frac{\gamma M_r}{r^2} \quad . \quad . \quad . \quad (1)$$

$$\rho C_v \frac{dT}{dt} = -(p_G + \tfrac{4}{3}aT^4)\frac{1}{\rho}\frac{d\rho}{dt} = \rho G - \frac{1}{r^2}\frac{d}{dt}(r^2 H) \quad . \quad (2)$$

In my original equations † I had erroneously taken the coefficient of aT^4 in equation (2) to be $\frac{1}{3}$ in place of $\frac{4}{3}$.

My general theory of stability remains unimpaired except for the numerical changes consequent on replacing $\frac{1}{3}$ by $\frac{4}{3}$ in equation (2). As before, two distinct conditions must be satisfied for a stellar structure to be stable.‡ The condition that a star shall not be liable to unstable contractions or expansions is absolutely unaffected by the error and remains, as before,

$$\beta + 3\alpha - n > -(\tfrac{7}{3} + \alpha)\frac{9s\lambda}{\lambda + 4 - 3s\lambda} \quad . \quad . \quad (3)$$

The condition that there shall be no explosive vibrations assumes the form

$$\beta + 3\alpha - n < (\tfrac{7}{3} + \alpha)\left(3 - \frac{1}{(\gamma - 1)(1 + \xi)}\right)\frac{\lambda}{\lambda + 4} \quad . \quad (4)$$

which must replace inequality (15) of my earlier paper.

* *Ast. Nach.*, 5545 (February 1928).
† *M.N.*, **85** (1925), 917.
‡ *M.N.*, **87** (1927), 401 and 720.

Both s and ξ depend on variations from the gas laws and vanish for purely gaseous stars. As we pass to stars of greater and greater mass, λ, the ratio of gas-pressure to pressure of radiation continually decreases, and finally vanishes for gaseous stars of infinite mass. As before, the range of values of $\beta + 3\alpha - n$ which are consistent with stability is limited at both ends, and for a purely gaseous star ($s = \xi = 0$) the length of range is

$$(\tfrac{7}{3} + \alpha)\left(\frac{3\gamma - 4}{\gamma - 1}\right)\frac{\lambda}{\lambda + 4} \quad . \quad . \quad . \quad (5)$$

This length of range vanishes altogether for a star of enormously great mass ($\lambda = 0$), and becomes very small indeed for massive known stars (*e.g.* for Plaskett's star λ is about 0·3 and $\lambda/(\lambda + 4) = 0{\cdot}07$). Thus the stability of stars of large mass shows either that they are not wholly gaseous, or else that $\beta + 3\alpha - n$ must be very nearly equal to zero, this being the value on which the two inequalities converge when λ is very small. The latter alternative appears to be ruled out by two distinct considerations.

In the first place it admits of no reasonable physical interpretation. The rate of generation of energy per unit mass of the star is supposed to be proportional to $\rho^\alpha T^\beta$. If energy is generated as the result of collisions or encounters of any kind, the rate of generation must contain a factor $\rho T^{\frac{1}{2}}$, so that $3\alpha + \beta$ is greater than 3·5, whereas n is only about $\frac{1}{2}$. Thus if $3\alpha + \beta - n$ is to be small, the generation of energy must be spontaneous, in which case $\alpha = \beta = 0$. Stability is now impossible, by condition (3), unless there are substantial deviations from the gas laws, and as condition (3) is precisely identical with that given in my former paper, the discussion proceeds as before.

Secondly, as a gaseous star contracts homologously, ρ varies as R^{-3}, where R is the radius of the star, while T varies as R^{-1}. The generation of energy G varies as $\rho^\alpha T^\beta$, and so as $R^{-(3\alpha+\beta)}$. The emission of energy E varies as T^n, and so as R^{-n}. Thus the ratio of emission to generation, E/G, varies as $R^{3\alpha+\beta-n}$, and if $3\alpha + \beta - n$ is very small, E/G remains almost unaltered by homologous contraction or expansion. Now in a configuration of equilibrium E/G = 1, and a star selects its position on a homologous series by choosing such a radius that E/G = 1. If E/G remains constant along the whole of a homologous series, as it would if $3\alpha + \beta - n$ were actually zero, the odds are infinite against this constant value being unity, so that it is infinitely improbable that a given star can find any configuration of equilibrium at all—so long, of course, as it remains purely gaseous. If $3\alpha + \beta - n$ is small but not actually zero, the argument takes the form that R has to change by enormous amounts to make any substantial change in E/G, and this is inconsistent with the small range in the observed radii of stars of moderate mass.

Either of these considerations seems adequate to exclude the alternative of gaseous stars. When deviations from the gas laws are admitted, the range of stability (5) is extended indefinitely. Condition (4), however, still requires that $\beta + 3\alpha - n$ shall be less than a

small quantity, and this requires an approximation to spontaneous generation of energy for which α and β both vanish, at least very approximately. The whole discussion now proceeds precisely as before. The effective condition which discriminates between stable and unstable configurations is condition (3), which remains unaltered from the original investigation.

The Rotation of the Galaxy. By J. S. Plaskett, F.R.S.

The theory of a rotation of the galactic system recently developed by Lindblad * in a series of papers constitutes an important and interesting hypothesis which serves to explain some of the observed stream and other motions in the stellar system. According to this hypothesis, the galactic system is composed of a number of sub-systems, each in approximately dynamical equilibrium, and each rotating about a common axis perpendicular to the galactic plane. The system with the highest speed of rotation, assumed to be that containing the Milky Way clouds, will be the most flattened towards the plane, will have a high star density and small residual velocities, so that the orbital motion will be nearly circular. Other sub-systems, all supposed to extend to about the same boundary in the plane, with a lower speed of rotation will be less flattened and have higher residual velocities. The globular cluster system, nearly spherical, is an example of a slowly rotating sub-system and, as is well known, has high residual velocities, while the sun and surrounding stars lie in a flattened system with a high speed of rotation. The asymmetry in stellar velocities observed by Stromberg † can be explained satisfactorily on this hypothesis, which requires the direction of the axis of asymmetry, $61°{\cdot}5 \pm 5°$ galactic longitude according to Stromberg, to be at right angles to the direction of the centre of rotation, hence also of the centre of the system of globular clusters which lies in longitude $325° \pm 3°$, the deviation being within the limits of error of the determinations. Kapteyn's stream motions are also explained, though not so directly, and the hypothesis, though still offering many difficulties, seems a very promising one.

The question has, however, been placed on a much firmer basis by two recent important papers by Oort ‡ and a third by Schilt, § which strongly confirm the presence of a rotational effect in the galaxy by an analysis of the observed radial velocities and proper motions of distant stars. The simple method of Oort for dealing with the radial velocities of distant galactic stars, which will be used in this paper, may be briefly summarised. Consider a star or group of stars at a distance r from the sun, and let the sun be at a distance R from the centre of the

* *Arkiv f. Mat.*, etc., Bd. 19A, Nos. 21, 27, 35; 19B, No. 7; and *M.N.*, **87**, 553.

† *Ap. J.*, **59**, 228 and **61**, 363.

‡ *B.A.N.*, No. 120, 132.

§ *Proc. Nat. Ac. Sc.*, **13**, 642.

galactic system, and suppose that $\frac{r}{R}$ is so small that terms of the second and higher orders may be neglected. The gravitational force K due to the whole system is obviously directed towards the centre and is some function of the distance R to the centre. The circular velocity of rotation at the sun will be

$$V = \sqrt{RK}.$$

It can be readily shown by a simple geometrical construction that the residual radial velocity, ρ', in the galactic plane caused by this rotation is

$$\rho' = rA \sin 2(l - l_0),$$

where l is the galactic longitude of the star, l_0 of the centre of the system, and where A has the value

$$A = \frac{V}{4R}\left(1 - \frac{R}{K}\frac{\delta K}{\delta R}\right).$$

"A" may be otherwise defined as the residual radial component, in kilometres per second due to the galactic rotation, for stars at a distance of one parsec. The relation may be stated more generally for stars at all latitudes by the expression

$$\rho = V_0 \cos \lambda + rA \sin 2(l - l_0) \cos^2 b,$$

where ρ is the radial velocity of the star, b its galactic latitude, V_0 the velocity of the sun towards, and λ the distance from, the solar apex for the group of stars considered. By means of this equation and a similar one for the proper motions, Oort was able to show that the distant stars exhibited a very definite residual effect which is most probably and simply explained as due to a rotation of the galactic system.

In view of the interest and importance of the matter, it seems worth while to present some further evidence on a galactic rotation. In this connection I am indebted to a suggestion by Dr. Schlesinger to observe here the radial velocities of a number of faint "*c*" stars, probably very suitable for testing the matter, and which is being undertaken. In the meantime, however, it seemed that the velocities of the fainter Bo to B5 stars, which are under investigation here by Mr. J. A. Pearce and the writer, would be useful in testing the rotational effect. While the investigation of the B stars, which includes all those of classes Bo to B5 brighter than visual magnitude 7·5 and north of − 11°, is still incomplete there are available, with the kind co-operation of Mr. Pearce, the final radial velocities of 250 Bo to B5 stars and the probable velocities of 140 more. With the addition of the velocities previously determined for the O to B5 stars, there are available, in the region of the sky within observational reach at Victoria, which includes the galactic region between longitudes from about 335° to 205°, the radial velocities of 610 O to B5 stars.

In the course of the investigation of the B stars, it has been found

that some 25 stars classified as B in the Henry Draper Catalogue are in reality absorption line O-type, and consequently, with this new material, it was decided to determine the rotation effect from both B and O stars. The positions of these 610 stars were transformed to galactic co-ordinates, and for this purpose and for obtaining the distance to the apex a set of graphs recently computed and drawn by Messrs. Pearce and Hill, and now being published, much reduced the labour. The positions were tabulated in the order of galactic longitude, and all stars whose galactic latitude was greater than 30°, some 45 in number, were rejected. The numbers were further reduced by the rejection of all O stars with a radial velocity greater than 60, and all B stars of a greater velocity than 50 km. per second. This seemed desirable in view of the relative smallness of the peculiar velocities, as one large velocity would have a greater effect in the solutions than five to ten average velocities. The question of group velocities in the B stars offered some difficulty, but it was finally decided, in view of the fact that the group motion is generally in the direction and of the order of the rotation effect, that only the Perseus Clusters need be considered. The velocities are high, about − 45 km. per second, and there are a considerable number in a small region. Finally rather more than half of these high-velocity group motion stars were rejected, leaving not more than one cluster star in the smaller class and magnitude groups and not more than two in the larger groups.

There remained for determining the galactic rotation the radial velocities of 65 O-type stars, of 132 Bo to B2 stars, and of 352 B3 to B5 stars, of which probably 90 per cent. are within 15° of the galactic plane and none at a greater distance than 30°. The distribution in galactic longitude between the limits observable here, 335° to 205°, is only moderately uniform, as the B stars are considerably less numerous between the longitudes of 80° and 150°. Further, the southerly longitudes are entirely unrepresented and the solutions are consequently subject to this disadvantage, partially compensated, however, by the fact that the longitudes available include the two positive maxima of the rotation effect at longitudes 10° and 190°, with one intervening negative maximum at longitude 100°.

From preliminary work, it was seen that the main factor for satisfactory results was a large number of stars in each group so as to reduce the effect of one or two large or discrepant peculiar velocities. Consequently, it was decided to use a relatively small number of groups in galactic longitude, eight being finally chosen, and to select these longitude regions so that the number of stars in each group rather than the longitude interval was nearly equal. The B-type stars were divided into two magnitude groups of nearly equal numbers, 243 stars $<6{\cdot}21$ vis. mag. and 241 $>6{\cdot}20$, and the O-type stars into 33 stars $<6{\cdot}51$ and 32 $>6{\cdot}50$. The B-type stars were also divided into two spectral divisions, thus forming two groups, bright and faint, of Bo to B2 stars and two groups of B3 to B5 stars, thus making four groups of B stars and two of O stars. The longitude intervals, the number of stars, the mean longitude, and the mean observed residual radial velocity are shown in the following tables :—

TABLE I.

Group Divisions of B Stars.

Limits of l.	Bo–B2 < 6·21.			Computed ρ'.		B3–B5 < 6·21.			Computed ρ'.	
	No.	l.	$\bar{\rho}'$.	l=325°.	l=332°·7.	No.	l.	$\bar{\rho}'$.	l=325°.	l=308°·5.
°		°					°			
347–24	2	6·0	+ 3·8	+7·8	+8·7	23	5·6	+3·7	+5·1	+5·2
26–42	4	34·5	+10·3	+6·0	+8·2	21	33·6	+2·7	+3·8	+1·7
43–64	6	49·2	+10·6	+3·5	+6·3	20	54·2	+1·7	+1·2	−1·7
65–80	10	71·3	− 3·7	−0·5	+1·5	14	70·9	−4·9	−1·0	−3·5
81–112	8	94·8	− 2·8	−2·9	−2·1	15	95·1	−2·2	−2·9	−3·5
113–149	7	130·0	+ 2·9	−0·3	−1·0	35	127·8	+0·3	−1·2	+0·8
150–171	12	161·8	+ 1·0	+5·4	+5·1	31	160·4	+5·9	+3·2	+5·2
172–193	17	177·0	+ 7·9	+7·3	+7·7	18	180·8	+6·0	+4·9	+5·5

Limits of l.	Bo–B2 > 6·20.			Computed ρ'.		B3–B5 > 5·20.			Computed ρ'.
	No.	l.	$\bar{\rho}'$.	l=325°.	l=328°·5.	No.	l.	$\bar{\rho}'$.	l=325°.
°		°					°		
347–24	1	24·0	+24·4	+14·6	+15·2	37	15·5	+3·7	+4·8
26–42	8	39·9	+12·6	+ 8·4	+ 9·8	31	31·2	+3·8	+3·3
43–64	11	51·4	0·0	+ 2·3	+ 0·9	23	50·0	+0·9	−0·1
65–80	11	71·7	− 5·7	− 8·7	− 7·8	26	73·3	−5·5	−4·8
881–112	16	100·4	−17·6	−16·0	−17·0	23	91·6	−6·9	−7·0
113–149	2	116·5	−12·2	−13·4	−15·4	13	129·2	−3·8	−2·0
150–171	2	163·5	+13·4	+10·0	+ 8·0	13	163·7	+3·6	+2·5
172–193	15	175·5	+12·1	+14·5	+13·2	9	178·4	+3·5	+4·4

TABLE II.

Group Divisions of O Stars.

Limits of l.	All O's.				O's < 6·51.				O's > 6·50.			
	No.	$\bar{l}$.	$\bar{\rho}'$.	Comp. ρ'. l=322°·3.	No.	$\bar{l}$.	$\bar{\rho}'$.	Comp. ρ'.	No.	$\bar{l}$.	$\bar{\rho}'$.	Comp. ρ'.
°		°				°				°		
334–357	6	340·7	+28·9	+15·7	3	344·0	+32·2	+15·2	3	337·3	+25·6	+13·5
24–46	8	38·4	+21·4	+13·5	2	35·0	+25·6	+15·5	6	39·5	+20·1	+15·7
48–67	11	58·2	+ 6·5	+ 2·2	9	60·2	+ 6·9	+ 6·3	2	49·0	+ 4·9	+ 8·9
70–92	7	76·0	−10·8	− 6·0	5	74·0	− 7·9	+ 1·5	2	81·0	−18·0	−13·3
103–117	9	110·0	−14·8	− 9·6	1	111·0	+ 9·3	− 2·0	8	110·0	−17·8	−16·7
137–170	7	154·1	+18·7	+12·4	3	157·7	+28·3	+13·1	4	151·5	+11·5	+ 9·2
174–177	12	175·0	+14·7	+20·9	6	175·7	+ 8·7	+18·2	6	174·3	+20·7	+23·2
190–206	5	197·4	+ 6·4	+21·4	4	199·2	+ 8·5	+18·9	1	190·0	− 3·6	+26·5

The additional columns headed "computed ρ'" will be referred to later. In obtaining the observed residual velocity from ρ, the radial velocity, the solar apex was taken as 271°, + 28°, the mean of Campbell's * and Wilson's † values. For the B-type stars a solar velocity of 20 km. per second was accepted, but, as it seemed likely to be larger for the O stars, a solution for K and V_0 was made from the 65 O stars, giving $V_0 = -22{\cdot}5$, $K = +8{\cdot}5$. If the seven high velocity O's previously rejected were included, the values became $V_0 = -27{\cdot}6$, $K = +5{\cdot}9$. However, as these stars were not used in the solutions for the rotation effect, the residual velocities of the O stars were obtained from the value of $V_0 = -22{\cdot}5$ km. per second.

The above longitude, type, and magnitude groups of B and O stars were then solved for the rotational effect, Oort's equation of condition becoming, as the stars are all close to the Galaxy and the $\cos^2 b$ term may be neglected,

$$K + \bar{r}A \sin 2(l - l_0) = \bar{\rho}'.$$

Solutions were made for the three unknowns, K, $\bar{r}$A, and l_0, and also, assuming $l_0 = 325°$, the longitude of the centre of the system of globular clusters, for K and $\bar{r}$A alone. The values of the unknowns with their probable errors are shown in the following table :—

TABLE III.

Values of l_0, K, and $\bar{r}$A.

Class and Mag. Limits.	No.	$\bar{m}$.	Solutions including l_0.			Solutions with $l_0 = 325°$.	
			l_0.	K.	$\bar{r}$A.	K.	$\bar{r}$A.
Bo–B2 < 6·21	66	5·00	332°·7±9°·8	+3·2±1·0	+ 5·9±0·9	+2·4±1·0	+ 5·4±1·4
Bo–B2 > 6·20	66	6·94	328·5±3·0	−0·4±0·9	+16·7±1·1	+0·2±0·8	+16·2±1·1
B3–B5 < 6·21	181	5·16	308·5±2·6	+0·9±0·3	+ 4·8±0·5	+1·1±0·6	+ 4·0±0·8
B3–B5 > 6·20	175	6·79	325·0±2·1	−1·2±0·3	+ 5·9±0·3	−1·2±0·2	+ 5·9±0·3
O all	65	6·43	322·3±7·2	+5·8±2·9	+17·1±4·0		
O < 6·51	33	5·55	..	..	..	+8·3±2·9	+11·2±4·5
O > 6·50	32	7·34	..	..	..	+4·2±2·0	+22·2±2·8

In addition, in order to see whether the relatively large range of longitude in the first, fifth, and sixth longitude groups introduced any systematic effect in the solutions, the group of B3 to B5 stars brighter than 6·21 mag., where the value for l_0 is discrepant, was rearranged into fifteen instead of eight longitude divisions, no one of which covered more than 15° of longitude. The solution gave practically identical values with that for the eight groups, l_0 being changed only 0°·2, thus justifying the small number of longitude groups.

The Values of l_0.—The values of l_0 obtained by the solutions are consistent except in the case of the brighter group of B3 to B5 stars,

* *A.S.P.*, **38**, 255.
† *Ast. J.*, **36**, 141.

where the value 308°·5 is more than 15° lower than the mean of the other groups. A comparison of the run of the mean residual velocities in this group in Table I. shows that the negative values begin and end at lower latitudes than the other groups, thus lowering the value of l_0. While the run of the residual velocities produces a low value of l_0, a comparison with the computed values of ρ' for $l_0 = 325°$, in the next column, shows that a change of two or three kilometres in the mean residual velocities at longitudes 50° and 130°, easily produced by the addition of two or three high-velocity stars, would bring the value of l_0 into agreement with the other groups. In view of this consideration, and of the further fact that for the brighter and nearer stars the rotational effect is necessarily not so definite, it does not seem unreasonable to consider this deviation as an accidental error due to chance combination of the peculiar velocities and to give it less weight than called for by its derived probable error. As a fair estimate the weight corresponding to a doubled probable error was given to this group, which even then makes its weight greater per star than that given to the allied group of the brighter Bo to B2 stars. Weighting the other four groups according to their probable errors, we have as the mean value of the longitude of the centre of rotation

$$l_0 = 324°{\cdot}5 \pm 1°{\cdot}8,$$

a value in almost exact agreement with the longitude, 325°, of the centre of the system of globular clusters and with Oort's * value of $324° \pm 2°$. The value obtained by Schilt † of 340° is based on the proper motions, hence of lower weight, with unknown systematic errors, of the Gyllenberg stars of the zone 35° to 40°, a narrow small circle of the sphere only crossing the galactic region at an angle of about 60°, at longitudes 40° and 140°. This value is obviously of lower weight than a determination from the radial velocities of the B stars, which are practically all in the galactic region.

It may hence be definitely concluded that the longitude of the centre of the galactic rotation, as determined from the radial velocities of the O and B stars, coincides with the longitude of the centre of the globular clusters, thus providing a confirmation of Lindblad's hypothesis of a common centre of rotation for the sub-systems of the Galaxy.

The Value of the K-term.—Only a few remarks need be made here about the value of the K-term, as it can be discussed more profitably when our investigation of the B stars is completed. It is certainly not a constant, seeming to be much larger for the hottest and brightest stars. It decreases from + 8 for the brighter to + 4 for the fainter O stars, from + 3 for the brighter to zero for the fainter Bo to B2 stars, and from + 1 for the brighter to − 1 for the fainter B3 to B5 stars. That this is not due to combining it in a solution with the rotation term can be seen from comparing the values thus obtained with those of the K-term derived from the residual velocities alone of the same B and O stars, where the variation is in the same direction but the values generally higher.

* *B.A.N.*, No. 120, p. 132.
† *Proc. Nat. Ac. Sc.*, **63**, 644.

TABLE IV.

The K-term from Residual Velocities.

Types.	Mag. Limits.	K.
Bo–B5	<5·5	+ 3·0
Bo–B5	5·5–6·65	+ 1·7
Bo–B5	>6·65	− 0·1
Bo–B2	All	+ 3·2
B3–B5	All	+ 0·8
O	<6·51	+10·8
O	>6·50	+ 6·2

While it is true that the velocities of the brighter B stars were determined at other observatories and of the fainter B's here, that does not apply to the O's nearly all determined at Victoria, where K has the greatest positive value and varies markedly, so that it can hardly be considered an instrumental or institutional effect. However, the question has not much bearing on the rotation effect as a solution, for $\bar{r}$A alone gave values differing little from those when K was included, and the discussion can well be postponed until fuller data are available.

The Rotation Term, $\bar{r}A$.—As will be seen at once from Table III., the rotation term $\bar{r}$A is always positive, and in the fainter and more distant stars is at least ten times its probable error. Further, for the stars of higher temperature especially, it increases markedly and nearly in the corresponding ratio as faintness and presumably increasing distance. Its increasing lower relative and absolute probable error in the more distant stars is to be expected as the rotation effect becomes the preponderating one and any irregular distribution of the peculiar velocities of the stars has proportionally less effect. Altogether the evidence given by these additional radial velocities of the B and O stars strongly supports the hypothesis of a rotation of the Galaxy. It seems worth while to redetermine the value of the constant A, the magnitude of the rotation effect in kilometres per second for stars at a distance of one parsec, from this new material. The following table, Table V., gives the computed value of A for the different groups of stars from the best available values of their mean parallaxes.

TABLE V.

The Value of A.

Class.	No.	$\bar{m}$.	Mean Parallax. Gerasimovic.	Oort.		Abs. Mag.	$\bar{\pi}$.	$\bar{r}$.	A.	π'.
			″	″		″	″			″
Bo–B2	66	5·00	·0019	·0030	..	·0032	·0027	460	·0117	·0035
Bo–B2	66	6·94	·0015	..	..	·0013	·0014	885	·0183	·0012
B3–B5	177	5·16	·0052	·0053	..	·0053	·0053	232	·0172	·0048
B3–B5	175	6·79	·0032	..	..	·0025	·0029	430	·0137	·0033
			Wilson.	Oort.	Plaskett.					
O	33	5·55	·0014	·0020	·0013	·0015	·0016	775	·0144	·0017
O	32	7·34	·0015	..	..	·0007	·0011	1130	·0196	·00086

In Table V. the first three columns contain the spectral class, the number of stars, and the average magnitude of the different groups. The next group of four columns contains various estimates of the mean parallax of the different groups. The last of these under the heading Abs. Mag. is the computed parallax on the assumption of an absolute magnitude of − 2·5 for Bo to B2, − 1·2 for B3 to B5, and − 3·5 for the O-types. This was added for the purpose of partially counteracting the effect of the less accurately determined proper motions of the faint stars, which are likely to give spuriously high values of their mean parallax. The next column, $\bar{\pi}$, is the simple mean of the previous estimates, while $\bar{r}$ is the average distance of the groups on the basis of Oort's * determination of $\bar{r}\bar{\pi} = 1{\cdot}24$ for the B-type stars and assumed the same for the O's. The next column gives the values of A for the different groups obtained by dividing the $\bar{r}$A's of Table III. by the $\bar{r}$ of the previous column. It was considered preferable to use the second set of values of $\bar{r}$A obtained by assuming $l_0 = 325°$, as if the peculiar distribution of the radial velocities of the B- and O-type stars is due to a rotation of the Galaxy, this rotation is probably about a common centre, and does not shift from longitude 332° to 308° as we go from bright B2 to bright B3 stars.

There is as satisfactory an accordance between the different values of A as can reasonably be expected from the uncertainties of the mean parallaxes and of the different values of $\bar{r}$A itself, and this agreement forms a decided confirmation of the reality of the rotational effect. The mean value of A obtained by weighting the individual values according to the number of stars in the group is

$$A = +{\cdot}0155 \pm {\cdot}0007 \text{ km. per sec. per parsec,}$$

and is about 18 per cent. less than Oort's † revised value of ·019, but as it is based on larger and more homogeneous material seems equally reliable, and in any case the difference is insignificant in view of the rather heterogeneous nature of Oort's data. Schilt's ‡ value of 0″·0106 or ·050 km./sec. is three times larger, but the relative unsuitability of proper motions for determining this constant and the non-galactic narrow zone of stars concerned make his determination one of smaller weight. As a matter of interest, what might be called the rotational values of the mean parallax of the various groups have been obtained from the mean value of A and the respective values of $\bar{r}$A, and are given in the last column of Table V. under π'. A comparison of these values with the otherwise determined mean parallaxes shows good agreement and indicates the usefulness of the rotational analysis in obtaining the distance of very remote objects. The differences between the rotational parallaxes and those determined from proper motions lie mainly in the groups of fainter stars in which the proper motions are uncertain. The parallaxes determined from the rotation term seem at least equally reliable as requiring a smaller dispersion in absolute magnitude as we go from bright to faint stars.

As a final check on the reality of the rotational effect, the computed values of the residual radial velocities have been entered in adjacent

* *B.A.N.*, **132**, 81. † *Ibid.*, **132**, 82. ‡ *Proc. Nat. Ac. Sc.*, **13**, 644.

columns to the observed $\bar{\rho}$'s in the various groups of B's and O's in Tables I. and II. The columns next to the observed velocities in Table I. and in the two magnitude subdivisions of Table II. have been computed with the values of $\bar{r}$A and K obtained, assuming $l_0 = 325°$, while the additional columns in the first three groups of the B's where the l_0 obtained in the solution differed more than 3° from 325° have been obtained by using the value of l_0, K, and $\bar{r}$A obtained in the complete solutions. In view of the nature of the material, and, in many cases, the small numbers of stars to balance any irregularities in the peculiar velocities, the agreement between the observed and computed velocities is remarkably good. This is particularly the case in the fainter and more distant stars where the rotational term becomes more pronounced, and where the probable errors of this term become so relatively small as to make the rotational effect very probable. Oort's * statement that his results "leave hardly any doubt as to the reality of this sin $2(l-325°)$ term." "It is possible, of course, that this term may be explained by systematic motions arising from another cause than rotation, but these systematic motions must bear a remarkably close resemblance to a rotation," is then emphatically strengthened by this investigation.

In conclusion, the probability of galactic rotation seems to be materially increased by this analysis of the radial velocities of the fainter B- and O-type stars. It will be of interest to see whether this conclusion is confirmed and strengthened when the remaining velocities of the B stars are completed, and when the velocities of a number of the faint and distant "*c*" stars are obtained.

The System of Procyon. By H. Spencer Jones, M.A., Sc.D., H.M. Astronomer.

In his work on *The Internal Constitution of the Stars*, Eddington remarks (p. 152), in reference to the testing of the mass-luminosity relation by observational data, that "Procyon is reckoned second-class because the semi-axis of the orbit is not well-determined; observations of this star are greatly to be desired and would form a most important check on the theory."

Three elliptic orbits have been computed for Procyon, the derived elements being given below for purposes of comparison.

	See.	Auwers.	Boss.
P	40*y.*	40·0*y.*	39·0*y.*
T	1891·0	1883·0	1886·5
e	0·45	0·227	0·324
a	0″·94	0″·956	1″·00
Ω	108°·3	267°	330°·7
ω	286°·35	72°·6	36°·8
i	33°·1	38°·3	14°·2
m_1/m_2	1 : 0·2	..	1 : 0·33

* *B.A.N.*, No. 120, 279.

See's orbit * is based on the series of micrometric measures of Procyon with respect to neighbouring stars made by Otto Struve † between 1851 and 1890, supplemented by the visual observations of the faint companion, discovered by Schaeberle in 1896, from 1896 to 1898.

Auwers' orbit ‡ is based on meridian observations of the bright star. Auwers makes the following remarks on the agreement between observation and the elements derived by him :—

"Die Darstellung der Rectascensionen ist wohl so gut wie nur verlangt werden kann ; die der Deklinationen erscheint hier nicht ganz befriedigend, die Elemente stimmen aber gut zu der Struve'schen Reihe von Declinationsanschlüssen, 1851–1890. Für den Positionswinkel des störenden Körpers geben diese Elemente 1897·0 297°·6, 1903·0 335°·9, die Entfernung des hellen Sterns vom Schwerpunct von 0″·93 bis 0″·84 abnehmend, während der beobachtete Begleiter in diesem Intervall den erheblich kleinern Bogen 321° bis 351° bei wachsender Distans (4″·7 bis 5″·2) durchlaufen hat. Diese Unterschiede sind so gross, dass es mir noch völlig zweifelhaft erscheint, ob in dem seit 1896 beobachteten Begleiter der störende Körper gefunden ist. Es werden deshalb einstweilen die oben angegebenen Elemente beizubehalten sein."

Boss's orbit is based on the meridian observations of the bright star and the micrometrical observations of the double star system from the date of discovery to 1905. Boss remarks that "The orbit is, of course, very uncertain, though the ratio of the masses is fairly well determined. The observation of the double is very difficult ; and the meridian-observations on a star so bright as this are very liable to be affected with anomalies." §

In view of the uncertainty attaching to some of the orbital elements and to the theoretical importance of the determination of the mass of Procyon, it has seemed worth while to make a redetermination of the orbital elements, using the available meridian and double-star observations. The faint companion is not now visible, and will not be visible for several years, so that within the next few years there is not likely to be much addition to the required data. Moreover, the determinations of the radial velocity of Procyon which have been made at the Cape help to determine the inclination of the orbit and the value of ω, the angle between the line of nodes and the major axis, with less uncertainty than is possible from the meridian and double-star observations alone.

I. *The Meridian Observations.*

The meridian observations of Procyon up to the year 1860 were discussed in Auwers' first memoir on the system of Procyon. ‖ This

* *A.J.*, **19**, 57, 1898.
† *Annals of the Pulkova Observatory*, vol. x.
‡ *A.N.*, No. 3929 (164, 299, 1903).
§ *Preliminary General Catalogue*, p. 267.
‖ Untersuchungen ueber veränderliche Eigenbewegungen. Erster Theil (Inaugural-Dissertation, Königsberg, 1862).

investigation was later supplemented by a further investigation * incorporating material up to about 1870, the whole of the material being discussed anew. In both investigations circular elements were derived. The method used was that adopted by Bessel † for the investigation of the variable proper-motion of Sirius. The right ascension of Procyon, derived from a certain series of observations, was compared with the tabular right-ascension according to Wolfer's "Tabulæ Reductionum." The right ascensions of the following six stars, α Tauri, β Orionis, α Orionis, β Geminorum, α Hydræ, and α Leonis, were compared with the corresponding tabular positions from Wolfer's Tables. The excess of the first difference over the second difference, designated q, is a quantity which oscillates backwards and forwards on account of the orbital motion of Procyon, and which can be analysed to give information as to the orbital elements. The mean position of the six reference stars is $7^h\ 6^m$, $+\ 7^\circ\ 59'$ (1900), that of Procyon being $7^h\ 34^m$, $+\ 5^\circ\ 29'$. The errors of a systematic nature of the form $\Delta\alpha_\alpha$ and $\Delta\alpha_\delta$ are therefore practically eliminated from the quantity q.

For the comparison in declination, it is not possible to choose a selection of well-observed fundamental stars whose mean declination is in close agreement with that of Procyon and whose mean right ascension does not greatly differ from the right ascension of Procyon. Auwers used the seven comparison stars, α Ceti, α Orionis, α Serpentis, γ Aquilæ, α Aquilæ, β Aquilæ, and α Aquarii, giving weight $\frac{2}{3}$ to each of the three stars in Aquila, as compared with unit weight for the remaining stars. The mean position of the comparison stars is $14^h\ 20^m$, $+\ 5^\circ\ 38'$ (1900); the declination agrees closely with that of Procyon, but the right ascension differs considerably. The excess of the difference between the observed declination of Procyon and the tabular value according to Wolfer's "Tabulæ Reductionum" over the weighted mean difference between the observed and tabular declinations of the comparison stars is called r. The derived values of r are practically independent of errors in the adopted latitude and refraction, but will be affected by systematic errors of the form $\Delta\delta_\alpha$.

The quantities q and r as derived from all available series of observations are affected by errors in the assumed positions and proper-motions of Procyon and of the comparison stars, and by the orbital motion of Procyon. The components of the orbital displacement at any time can be expressed in the form :

$$\Delta\alpha = -g\,e + g\,\cos u + h\,\sin u,$$
$$\Delta\delta = -g'e + g'\cos u + h'\sin u,$$

where

$$g = \tfrac{1}{1\overset{s}{.}5}\,a\sec\delta\,(\cos\omega\sin\Omega + \sin\omega\cos\Omega\cos i),$$
$$h = -\tfrac{1}{1\overset{s}{.}5}a(1-e^2)^{\frac{1}{2}}\sec\delta\,(\sin\omega\sin\Omega - \cos\omega\cos\Omega\cos i),$$
$$g' = a\,(\cos\omega\cos\Omega - \sin\omega\sin\Omega\cos i),$$
$$h' = -a(1-e^2)^{\frac{1}{2}}\,(\sin\omega\cos\Omega + \cos\omega\sin\Omega\cos i),$$

* *Monatsberichte der Akad. d. Wiss. zu Berlin*, p. 362, 1873.
† *A.N.*, Nos. 745–8, 1851.

and in which a denotes the semi-axis major of the orbit of the bright star about the centre of gravity of the system (expressed in seconds of arc), e is the eccentricity of the system, ω is the angle between the line of nodes and the major axis, Ω is the position angle of the line of nodes, u is the eccentric anomaly, and i is the inclination of the orbit plane. $\Delta\alpha$, $\Delta\delta$ are supposed to be expressed in seconds of time and seconds of arc respectively.

Including corrections to the tabular positions and proper-motions, the values of q and r are therefore expressible in the form :

$$q = a + b\,(t - 1850) + g\,\cos u + h\,\sin u,$$
$$r = a' + b'(t - 1850) + g'\cos u + h'\sin u,$$

t denoting the time measured from the epoch 1850.

From analysis of the derived values of q and r, the quantities a, b, a', b', g, h, g', h', as well as e, the eccentricity of the orbit, P the period and T the time of periastron, may be determined. If we write *

$$a(\cos\omega\cos\Omega - \sin\omega\sin\Omega\cos i) = \mathrm{P},$$
$$a(\sin\omega\cos\Omega + \cos\omega\sin\Omega\cos i) = \mathrm{Q},$$
$$a(\cos\omega\sin\Omega + \sin\omega\cos\Omega\cos i) = \mathrm{R},$$
$$a(\sin\omega\sin\Omega - \cos\omega\cos\Omega\cos i) = \mathrm{S},$$

then

$$\mathrm{P} = g'\,;\ \mathrm{Q} = -h'(1-e^2)^{-\frac{1}{2}};\ \mathrm{R} = 15g\cos\delta\,;\ \mathrm{S} = -15h\cos\delta(1-e^2)^{-\frac{1}{2}},$$

and

$$\tan(\omega - \Omega) = \frac{\mathrm{Q} - \mathrm{R}}{\mathrm{P} + \mathrm{S}},$$
$$\tan(\omega + \Omega) = \frac{\mathrm{Q} + \mathrm{R}}{\mathrm{P} - \mathrm{S}},$$

determine ω and Ω, whilst

$$a^2\cos i = \mathrm{QR} - \mathrm{PS},$$
$$a^2(1 + \cos^2 i) = \mathrm{P}^2 + \mathrm{Q}^2 + \mathrm{R}^2 + \mathrm{S}^2,$$

determine a and $\cos i$.

The most convenient procedure for the determination of g, h, g', h', (from which a, i, ω, and Ω can be derived), e, P (or n the mean motion), and T is to assume preliminary values for the elements, to compute the resulting values of q and r, and to use the residuals between the observed and computed values for the differential correction of the elements. Thus we have

$$dq = da + (t - 1850)\,.\,db + dg\,.\cos u + dh\,.\sin u + (h\cos u - g\sin u)du$$

and

$$u - e\sin u = n(t - \mathrm{T}) \times \frac{2\pi}{360},$$

where n is expressed in degrees per year and $(t - \mathrm{T})$ is expressed in years.

* The use of P in these expressions will not lead to confusion with the orbital period.

Therefore

$$du - e\cos u\,.\,du - \sin u\,.\,de = (t - \mathrm{T})\frac{2\pi}{360}\,.\,dn - n\,.\,\frac{2\pi}{360}d\mathrm{T}.$$

Whence

$$dq = da + (t - 1850)db + \cos u\,.\,dg + \sin u\,.\,dh + \lambda\sin u\,.\,de + \lambda(t - \mathrm{T})\frac{2\pi}{360}\,.\,dn - \lambda n\,.\,\frac{2\pi}{360}d\mathrm{T},$$

where

$$\lambda = \frac{h\cos u - g\sin u}{1 - e\cos u},$$

and similarly for dr.

The equations of conditions are most conveniently used in the forms :

$$\mathrm{A} + \frac{t - 1850}{50}\mathrm{B} + \cos u\,.\,\mathrm{G} + \sin u\,.\,\mathrm{H} + \lambda\sin u\,.\,\mathrm{E} + \lambda\,.\,\frac{t - \mathrm{T}}{100}\mathrm{N} - \lambda\,.\,\frac{n}{10}\tau = 15dq(\mathrm{O} - \mathrm{C}),$$

$$\mathrm{A}' + \frac{t - 1850}{50}\mathrm{B}' + \cos u\,.\,\mathrm{G}' + \sin u\,.\,\mathrm{H}' + \lambda'\sin u\,.\,\mathrm{E} + \lambda'\,.\,\frac{t - \mathrm{T}}{100}\mathrm{N} - \lambda'\,.\,\frac{n}{10}\tau = dr(\mathrm{O} - \mathrm{C}),$$

wherein

$\mathrm{A} = 15da,$	$\mathrm{G} = 15dg,$	$\mathrm{E} = de,$
$\mathrm{B} = 750db,$	$\mathrm{H} = 15dh,$	$\mathrm{N} = \frac{5}{9}\pi\,.\,dn,$
$\mathrm{A}' = da',$	$\mathrm{G}' = dg',$	$\tau = \frac{1}{18}\,.\,\pi d\mathrm{T}.$
$\mathrm{B}' = 50db',$	$\mathrm{H}' = dh',$	

The equations of condition are solved by least squares for the corrections to the adopted values.

In the present investigation, the method used by Auwers has been followed because it eliminates to a large extent the systematic errors inherent in each particular series of observations, and avoids the necessity of investigating such errors, and because, in addition, the material used by Auwers in his earlier investigations can be utilised. Much of this material is derived from annual volumes of results which were not collected into the usual catalogue form. The use of the material discussed by Auwers has enabled a much larger volume of observations to be used than would have been possible otherwise, since many of the early series of observations are lacking in the library of this observatory. The most important error which has not been eliminated is the systematic error in declination, dependent upon right ascension. For catalogues whose systematic errors are given by Boss in the *Preliminary General Catalogue*, the error of this nature could have been eliminated, but for other series of observations special investigations would have been necessary. The gain in accuracy would not have been commensurate with the labour involved. No corrections have therefore

been applied to any of the observations used on account of systematic errors of the form $\Delta\delta_\alpha$. Such errors will have the effect of accidental errors in the discussion of the whole material.

The positions and proper-motions from Boss's catalogue have been adopted as a basis throughout, and values of q and r which have been taken from the two investigations of Auwers have been corrected for the differences in the positions and proper-motions of the stars according to Wolfer's "Tabulæ Reductionum" and according to Boss's catalogue.

In the reductions from apparent to mean places of the stars, different values of the constants of nutation and of aberration have been employed at different times. It would be desirable, if it were possible, to reduce all the observations to the same values for these constants. In the case of aberration, this could only be effected by correcting each individual observation. The maximum correction is, however, only of the order of a few hundredths of a second of arc in general; in forming q and r and taking the mean of observations at different times of the year the mean correction is very much reduced and can with safety be neglected, and any errors arising from the use of different values of the constant of aberration at different times can be treated as accidental errors.

Nutation is a long-period term, and it is therefore important that the observational data should be reduced to a uniform value of the constant of nutation. The value adopted is the Paris Conference value of $9''{\cdot}210$, used in the ephemerides from 1901 onwards. The values used previously were in general either Bessel's value ($9''{\cdot}250$) or Peters's value ($9''{\cdot}224$). It is easily shown that if a value of $(9''{\cdot}210 - \eta)$ for the constant of nutation is used in the reductions, the corrections to be applied to the derived values of q and r to reduce to the basis of the value $9''{\cdot}210$ are

$$\Delta q = -\eta(+\cdot0019 \sin\Omega + \cdot0001 \cos\Omega) \text{ in seconds of time,}$$
$$\Delta r = -\eta(+\cdot540 \sin\Omega + 1\cdot164 \cos\Omega) \quad ,, \quad ,, \quad \text{arc,}$$

Ω denoting the mean longitude of the Moon's ascending node.

The correction Δq is too small to take into account, but the correction Δr must be applied. It is sufficient to use the value of Ω corresponding to the mean epoch of observation and to apply the correction to the mean value of r resulting from all the observations.

An extensive search was made in the library of the observatory for suitable series of observations in order to ensure that the material used should be to all intents and purposes complete up to date. To avoid overlooking any important catalogue or series of observations, reference was made to the list prepared by Auwers,* which was supplemented by the inclusion of later material. In addition, Auwers' earlier investigations of the system of Procyon were examined and such material as had not already been utilised was incorporated. Where later reductions of material used by Auwers were available, *e.g.* the reduction

* "Abgekürtze Bezeichnungen für Sterncatalogue," *A.N.*, No. 4176, Bd. 174, 1907.

of Bradley's observations by Auwers, Downing's reduction of Taylor's observations at Madras, Halm's reduction of Henderson's observations at Edinburgh, the later reductions were utilised.

In some instances results have been taken from annual volumes of observations and combined into means for periods of two or three years. This may have been done for one of several reasons : (1) The results in question may not have been combined into catalogue form, *e.g.* Königsberg observations, 1820 onwards ; Cambridge observations, 1836 onwards ; Radcliffe observations, 1862–79,* etc. (2) The catalogue may not give the mean epochs of observation, *e.g.* Pond's Greenwich catalogue of 1112 stars for 1830. (3) The catalogue may be based upon observations which cover a much longer period of time than is desirable for combining into a single mean, *e.g.* the three Paris catalogues, (*a*) Paris, 1845, from observations 1837–1853 ; (*b*) Paris, 1860, from observations 1854–1867 ; (*c*) Paris, 1875, from observations 1868–1881. (4) Even though the observations upon which the catalogue is based only cover a period of several years, provided that the observations are sufficiently numerous and well distributed, it is preferable to divide the period covered by the observations into two or more periods. This is only possible when the individual observations are readily accessible in the form of annual catalogues or otherwise.

The results of some series of observations are given as corrections to the positions of Auwers' Fundamental Catalogue, the position of Procyon being reduced to the centre of gravity by means of the ephemeris given by Auwers. In such cases the results have been reduced to Boss by correcting for the difference between the positions of Auwers and Boss at the epoch in question, and the position of Procyon has been reduced back to that of the bright star. In the more recent series of observations it is not always clear exactly how the position given for Procyon has been derived, especially when the reduction from apparent to mean place has been taken from the 10-day ephemeris in one of the almanacs. It may be mentioned here that the method of dealing with the orbital motion in the almanacs has been by no means uniform. It is suspected that, in some cases, observations of Procyon have been reduced to the mean place for the beginning of each year by means of an almanac ephemeris, whereby the orbital motion from the beginning of the year is taken out. The final mean place has then been derived from these annual means, but it does not correspond, as far as orbital motion is concerned, with the mean epoch of observation. Every case of doubt has been checked as far as is possible.

The values of q and r as used for the subsequent discussion are collected in Tables I. and II. respectively. In these tables, the first column gives the name of the observatory, the second the period covered by the observations, the third the mean epoch of observation, the fourth the values of q and r respectively, and the fifth the assigned weight. The sixth column gives the values of q and r computed from

* These observations have been reduced to catalogue form by Hans Osten, *Nova Acta, Abh. der Kaiserl. Leop. Carol. Deutschen Akad. der Naturforscher* (Halle), Bd. xcii., No. 1, 1910. This catalogue was not available for consultation.

the final elements, and the seventh gives the residual between the observed and computed values.

The weights given in these tables are assigned in a manner which is necessarily to a large extent arbitrary. Such factors as the number of observations, the period covered by them, the type of instrument used, and the method of observing affect the weights. These factors have as far as possible been taken into consideration as well as the relative weights of observations for certain series which are given in Appendix III. of Boss's Catalogue. Although another investigator would doubtless assign weights in a manner differing considerably from the present system, the weighting adopted should undoubtedly suffice to allow for the grosser inequalities in weight.

TABLE I.

Values of q.

Observatory.	Period of Observation.	Mean Epoch.	Observed *q*.	Weight.	Computed *q*.	O−C.
			s		s	s
Greenwich	1755–62	1757·2	−·048	4·0	−·050	+·002
,,	1765–68	67·2	+·060	1·0	+·038	+·022
,,	1769–72	70·6	+·072	1·0	+·065	+·007
,,	1775–76	75·9	+·029	0·5	+·063	−·034
,,	1780–84	82·4	−·022	0·5	+·021	−·043
,,	1785–86	85·9	−·031	0·5	−·006	−·025
,,	1795–96	95·9	−·087	0·5	−·056	−·031
,,	1797–01	99·4	−·080	0·5	−·052	−·028
Palermo	1803–05	1804·9	+·081	0·0	..	..
Greenwich	1803–06	05·4	−·011	1·0	+·004	−·015
,,	1807–10	08·9	+·050	1·0	+·046	+·004
Königsberg	1813–19	15·0	+·068	3·0	+·059	+·009
Greenwich	1811–19	18·3	+·032	3·5	+·044	−·012
Dorpat	1818–22	20·4	−·035	1·0	+·031	−·066
Königsberg	1820–24	22·3	+·016	5·0	+·016	·000
Dublin	1822–24	23·5	+·013	1·0	+·008	+·005
Dorpat	1822–26	24·8	+·009	3·0	−·002	+·011
Königsberg	1825–31	28·2	−·030	6·0	−·027	−·003
Åbo	1827–30	29·0	−·027	6·5	−·033	+·006
St. Helena	1829–33	31·5	−·054	3·0	−·048	−·006
Cambridge	1829–35	32·3	−·048	4·0	−·053	+·005
Greenwich	1829–35	32·3	−·072	4·0	−·053	−·019
Cape	1832–33	32·8	−·060	2·0	−·056	−·004
Königsberg	1832–35	33·4	−·052	5·0	−·057	+·005
Madras	1831–42	33·7	−·045	2·5	−·058	+·013
San Fernando	1834–35	34·8	−·064	3·0	−·062	−·002

TABLE I.—*continued.*

Observatory.	Period of Observation.	Mean Epoch.	Observed q. s	Weight.	Computed q. s	O−C. s
Cape	1834–40	1836·1	−·060	4·0	−·064	+·004
Cambridge	1836–39	38·0	−·068	3·0	−·065	−·003
Greenwich	1836–39	38·1	−·079	3·0	−·065	−·014
Königsberg	1836–40	38·6	−·099	3·5	−·064	−·035
Brussels	1838–39	38·7	−·049	0·5	−·064	+·015
Washington	1838–42	40·5	−·041	1·5	−·057	+·016
Edinburgh	1835–45	41·1	−·062	10·0	−·054	−·008
Cambridge	1840–43	41·9	−·024	2·5	−·047	+·023
Greenwich	1840–43	42·2	−·050	3·0	−·044	−·006
Königsberg	1841–44	42·9	−·045	4·0	−·038	−·007
Geneva	1842–45	43·8	−·053	4·0	−·028	−·025
Königsberg	1843–44	43·9	−·026	1·5	−·028	+·002
Madras	1843–47	45·0	−·005	1·5	−·013	+·008
Paris	1837–53	45·3	−·008	5·0	−·009	+·001
Radcliffe	1840–53	45·4	+·023	3·0	−·007	+·030
Pulkova	1842–49	45·5	+·010	12·0	−·005	+·015
Cambridge	1844–47	45·8	+·019	2·5	−·001	+·020
Greenwich	1844–47	45·9	−·056	3·0	·000	−·056
Edinburgh	1845–47	46·3	+·002	2·0	+·003	−·001
Königsberg	1845–49	47·3	+·041	4·0	+·018	+·023
Washington	1846–50	47·8	+·012	6·0	+·024	−·012
Geneva	1846–49	47·9	+·045	4·0	+·025	+·020
Washington	1846–50	48·1	−·008	6·0	+·027	−·035
Greenwich	1848–50	49·3	+·013	3·0	+·039	−·026
Cambridge	1848–51	49·8	+·027	2·0	+·043	−·016
Madras	1848–52	50·2	+·050	1·5	+·046	+·004
Santiago	1850–52	51·4	+·042	1·0	+·052	−·010
Washington	1851–52	51·6	+·015	2·0	+·053	−·038
Geneva	1850–53	51·8	+·067	4·0	+·053	+·014
Greenwich	1851–53	52·0	+·068	3·0	+·054	+·014
Cape	1849–52	52·2	+·140	1·0	+·054	+·086
Edinburgh	1850–54	52·2	−·005	1·0	+·054	−·059
Cambridge	1852–55	53·8	+·051	2·0	+·054	−·003
Paris	1854–55	54·6	+·044	2·0	+·052	−·008
Greenwich	1854–56	55·2	+·049	3·5	+·051	−·002
Geneva	1854–56	55·5	+·049	3·0	+·050	−·001
Edinburgh	1855–57	55·7	+·011	2·5	+·049	−·038
Brussels	1855–56	56·3	+·049	0·5	+·048	+·001

TABLE I.—*continued.*

Observatory.	Period of Observation.	Mean Epoch.	Observed q. s	Weight.	Computed q. s	O−C. s
Paris	1856–57	1857·0	+·032	3·0	+·044	−·012
Radcliffe	1854–61	57·3	+·065	2·0	+·042	+·023
Cambridge	1856–58	57·4	+·052	1·5	+·041	+·011
Geneva	1857–59	57·8	+·036	2·0	+·040	−·004
Königsberg	1857–59	57·9	+·067	1·0	+·040	+·027
Greenwich	1857–58	57·9	+·030	2·5	+·040	−·010
Edinburgh	1858–60	58·7	+·033	2·5	+·035	−·002
Paris	1858–59	58·9	+·024	3·0	+·034	−·010
Cambridge	1859–60	59·2	+·012	0·5	+·032	−·020
Cape	1856–61	59·6	+·020	2·5	+·029	−·009
Greenwich	1859–60	59·9	+·016	2·0	+·027	−·011
Paris	1860–61	60·8	+·015	3·0	+·021	−·006
Melbourne	1860–63	62·1	+·022	1·5	+·012	+·010
Edinburgh	1861–63	62·4	−·002	3·0	+·010	−·012
Greenwich	1861–63	62·6	+·026	3·0	+·008	+·018
Radcliffe	1862–64	62·8	+·033	1·0	+·006	+·027
Paris	1862–63	62·9	+·007	2·5	+·005	+·002
Washington	1861–63	62·9	+·023	3·5	+·005	+·018
Cape	1861–64	63·1	+·017	3·0	+·004	+·013
Cambridge	1862–65	64·4	−·028	3·0	−·005	−·023
Paris	1864–66	65·0	·000	2·5	−·010	+·010
Greenwich	1864–65	65·0	−·022	3·5	−·010	−·012
Washington	1864–65	65·1	+·017	3·5	−·011	+·028
Edinburgh	1864–66	65·1	−·014	2·5	−·011	−·003
Pulkova	1861–72	65·5	+·038	9·0	−·013	+·051
Brussels	1857–78	66·1	−·027	3·5	−·018	−·009
Cape	1865–67	66·4	+·028	3·0	−·020	+·048
Radcliffe	1865–67	66·7	−·065	1·0	−·022	−·043
Washington	1866–67	66·7	+·001	4·0	−·022	+·023
Greenwich	1866–67	66·9	−·026	3·5	−·024	−·002
Washington	1866–67	66·9	−·002	3·5	−·024	+·022
Melbourne	1863–70	67·2	−·017	3·0	−·027	+·010
Cambridge	1866–69	67·7	−·037	3·0	−·030	−·007
Paris	1867–68	67·9	−·011	2·5	−·032	+·021
Glasgow	1860–81	68·0	−·041	4·0	−·033	−·008
Harvard	1867–68	68·0	−·008	1·0	−·033	+·025
Edinburgh	1867–69	68·1	−·036	2·0	−·034	−·002
Washington	1868–69	68·7	−·035	3·0	−·037	+·002

TABLE I.—*continued.*

Observatory.	Period of Observation.	Mean Epoch.	Observed q.	Weight.	Computed q.	O−C.
			s		s	s
Cape	1868–70	1868·7	−·018	3·0	−·037	+·019
Washington	1868–70	69·3	−·045	4·0	−·042	−·003
Radcliffe	1868–69	69·4	−·038	1·0	−·043	+·005
Paris	1869–70	70·1	−·046	2·5	−·046	·000
Radcliffe	1870	70·4	−·050	1·5	−·049	−·001
Madras	1862–87	70·7	−·047	3·0	−·051	+·004
Radcliffe	1871–72	71·6	−·046	1·5	−·055	+·009
Berlin	1869–73	71·9	−·057	4·5	−·057	·000
Greenwich	1868–76	71·9	−·066	9·0	−·057	−·009
Harvard	1871–72	72·2	−·075	1·0	−·058	−·017
Paris	1871–74	72·6	−·049	3·0	−·060	+·011
Washington	1871–73	72·7	−·068	4·0	−·061	−·007
Cape	1871–79	74·4	−·047	5·0	−·068	+·021
Harvard	1870–79	74·7	−·095	3·0	−·069	−·026
Washington	1874–75	74·8	−·066	4·0	−·069	+·003
Radcliffe	1873–76	75·5	−·079	1·5	−·071	−·008
Paris	1875–78	76·3	−·059	3·0	−·072	+·013
Pulkova	1874–80	76·7	−·081	8·0	−·073	−·008
Sydney	1877–81	78·4	−·102	1·0	−·073	−·029
Melbourne	1871–85	78·4	−·051	4·0	−·073	+·022
Greenwich	1877–86	81·6	−·061	9·0	−·059	−·002
Cape	1879–85	82·0	−·073	5·0	−·056	−·017
Paris	1881–82	82·0	−·037	4·0	−·056	+·019
,,	1883–84	83·9	−·029	6·0	−·038	+·009
Pulkova	1880–87	84·3	−·030	10·0	−·033	+·003
Radcliffe	1880–93	85·1	−·032	3·5	−·022	−·010
Strasbourg	1884–88	85·5	−·023	7·0	−·018	−·005
Paris	1885–87	86·3	−·009	5·5	−·005	−·004
Cape	1886–87	86·5	+·025	3·0	−·003	+·028
Greenwich	1887–88	87·7	+·046	1·5	+·013	+·033
Pulkova	1881–94	88·1	+·002	4·0	+·017	−·015
Glasgow	1886–92	88·4	−·051	0·5	+·020	−·071
Lisbon	1887–90	88·7	+·036	3·0	+·022	+·014
Cape	1887–91	88·9	+·024	3·0	+·024	·000
Washburn	1888–90	89·2	+·051	0·5	+·027	+·024
Paris	1888–90	89·4	+·029	5·5	+·029	·000
Melbourne	1885–94	89·6	+·014	5·0	+·032	−·018
Greenwich	1889–91	90·7	+·030	1·5	+·039	−·009

TABLE I.—*continued.*

Observatory.	Period of Observation.	Mean Epoch.	Observed q. s	Weight.	Computed q. s	O—C. s
Strasbourg	1888–93	1891·2	+·050	4·0	+·042	+·008
Paris	1891–92	91·8	+·065	5·5	+·044	+·021
Greenwich	1892	92·5	+·050	1·5	+·046	+·004
Nice	1888–96	92·9	+·055	4·0	+·046	+·009
Paris	1893	93·4	+·043	4·5	+·047	−·004
Greenwich	1893	93·5	+·048	4·0	+·047	+·001
Berlin	1892–97	94·1	+·043	0·5	+·046	−·003
Greenwich	1894	94·5	+·048	2·0	+·045	+·003
,,	1895	95·5	+·054	3·0	+·043	+·011
,,	1896	96·5	+·047	3·0	+·039	+·008
,,	1897–98	97·9	+·031	4·0	+·033	−·002
Pulkova	1894–02	98·0	+·027	12·0	+·032	−·005
Radcliffe	1894–03	98·6	+·041	1·5	+·029	+·012
Paris	1897–99	98·8	+·035	4·0	+·027	+·008
Nice	1897–00	98·9	+·019	3·5	+·026	−·007
Greenwich	1899–00	99·8	+·034	6·0	+·021	+·013
Odessa	1899–03	1901·0	+·020	5·0	+·013	+·007
Paris	1900–02	01·4	+·002	4·0	+·011	−·009
Greenwich	1901–02	01·8	+·006	6·0	+·008	−·002
,, (alt.)	1899–08	03·3	−·003	3·0	−·003	·000
,,	1903–05	04·1	−·012	6·0	−·012	·000
Odessa	1904–07	05·0	−·033	5·0	−·016	−·017
Pulkova	1903–06	05·0	−·010	6·0	−·016	+·006
Paris	1903–06	05·3	−·030	3·5	−·018	−·012
Königsberg	1901–07	06·5	−·032	1·0	−·027	−·005
Pulkova	1906–09	07·0	−·029	6·0	−·031	+·002
Cape	1905–11	07·0	−·049	3·0	−·031	−·018
Washington	1903–11	07·2	−·010	1·0	−·033	+·023
Greenwich	1906–08	07·5	−·032	3·5	−·035	+·003
Lick	1907–08	08·0	−·054	0·5	−·038	−·016
Greenwich	1909–13	10·4	−·057	1·0	−·054	−·003
,,	1909–11	10·7	−·053	3·5	−·057	+·004
Abbadia	1900–14	11·0	−·053	4·0	−·058	+·005
Paris	1909–14	11·7	−·082	8·0	−·062	−·020
Cape	1912–16	12·8	−·069	2·0	−·068	−·001
Greenwich	1912–13	13·0	−·089	3·0	−·069	−·020
Pulkova	1911–18	14·0	−·082	12·0	−·074	−·008
Greenwich	1915–21	18·2	−·087	12·0	−·081	−·006
Abbadia	1919–21	21·0	−·101	2·0	−·073	−·028
Cape	1918–25	22·0	−·068	2·0	−·069	+·001

TABLE II.

Values of r.

Observatory.	Period of Observation.	Mean Epoch.	Observed *r*.	Weight.	Computed *r*.	O−C.
Greenwich	1755–62	1756·6	+″·19	1·0	−″·24	+″·43
,,	1812–13	1813·1	− ·45	1·0	− ·47	+ ·02
Königsberg	1813–19	14·8	−2·10	0·5	− ·71	−1·39
Greenwich	1814–18	16·2	− ·71	3·0	− ·89	+ ·18
Göttingen	1820	20·3	−1·67	0·5	−1·21	− ·46
Königsberg	1820–21	20·9	−1·49	1·5	−1·14	− ·35
,,	1822–25	23·9	−1·32	3·0	−1·32	·00
Turin	1823–25	24·6	−1·64	0·5	−1·31	− ·33
Dorpat	1822–26	24·8	−1·54	4·0	−1·31	− ·23
Greenwich	1825–28	26·7	−1·44	2·0	−1·26	− ·18
Speyer	1826–27	26·8	−1·09	1·5	−1·26	+ ·17
Königsberg	1826–28	27·4	−1·08	2·5	−1·23	+ ·15
Modena	1828	28·5	−1·45	0·5	−1·17	− ·28
Åbo	1827–30	29·0	−1·25	4·5	−1·14	− ·11
Königsberg	1829–32	30·9	−1·02	2·5	−1·01	− ·01
St. Helena	1829–33	31·5	−1·04	2·0	− ·96	− ·08
Greenwich	1829–35	32·3	− ·87	3·5	− ·87	·00
Madras	1831–42	32·7	−1·51	1·5	− ·83	− ·68
Cape	1832–33	32·8	− ·90	1·5	− ·82	− ·08
Cambridge	1829–35	33·0	−1·17	1·5	− ·80	− ·37
Münich	1833–34	33·8	− ·32	1·5	− ·71	+ ·39
Königsberg	1833–35	34·4	− ·66	2·0	− ·65	− ·01
Cape	1834–40	35·1	−1·49	3·0	− ·57	− ·92
Greenwich	1836–39	37·6	− ·48	2·0	− ·23	− ·25
Edinburgh	1835–45	37·9	− ·03	5·0	− ·19	+ ·16
Cambridge	1836–39	38·0	− ·46	1·5	− ·18	− ·28
Greenwich	1836–39	38·1	+ ·09	2·0	− ·17	+ ·26
Königsberg	1836–40	38·6	− ·48	2·0	− ·15	− ·33
Cambridge	1840–43	41·9	− ·03	1·5	+ ·28	− ·31
Greenwich	1840–43	42·2	+ ·42	2·0	+ ·30	+ ·12
Pulkova	1842–49	42·2	+ ·59	3·0	+ ·30	+ ·29
Königsberg	1841–43	42·4	+ ·70	2·0	+ ·33	+ ·37
Geneva	1842–45	43·8	+ ·41	2·0	+ ·45	− ·04
Königsberg	1843–44	43·9	+ ·74	1·5	+ ·45	+ ·29
Madras	1843–47	45·0	+ ·51	3·0	+ ·47	+ ·04
Radcliffe	1840–53	45·2	+ ·81	1·0	+ ·48	+ ·33
Paris	1837–53	45·3	+ ·29	5·0	+ ·48	− ·19
Cambridge	1844–46	45·3	+ ·44	1·0	+ ·48	− ·04

TABLE II.—*continued.*

Observatory.	Period of Observation.	Mean Epoch.	Observed r.	Weight.	Computed r.	O−C.
			″		″	″
Greenwich	1844–47	1845·7	+·61	2·0	+·47	+·14
Königsberg	1844–47	45·9	+1·06	1·5	+·46	−·60
Washington	1846–50	47·3	+·62	1·0	+·36	+·26
Geneva	1846–49	47·9	+1·64	0·5	+·31	+1·33
Washington	1846–50	48·1	+·23	1·5	+·29	−·06
Königsberg	1848–49	49·0	+·57	1·5	+·14	+·43
Greenwich	1848–50	49·6	+·31	2·0	+·06	+·25
Königsberg	1848–53	49·7	+·80	1·0	+·05	+·75
Madras	1848–52	50·0	+·27	3·0	−·02	+·29
Santiago	1850–52	51·4	−·82	0·5	−·24	−·58
Geneva	1850–53	51·8	+·04	0·5	−·31	+·35
Greenwich	1851–53	52·0	−·46	2·5	−·34	−·12
Königsberg	1850–54	52·8	+·77	0·5	−·49	+1·26
Berlin	1853–54	53·8	−·92	1·0	−·64	−·28
Paris	1854–55	54·4	−1·25	0·5	−·69	−·56
Greenwich	1854–56	55·2	−·66	4·0	−·83	+·17
Geneva	1854–56	55·5	−·52	1·0	−·87	+·35
Radcliffe	1854–61	55·8	−·25	1·0	−·90	+·65
Paris	1856–57	56·8	−1·00	2·0	−1·01	−·01
Geneva	1857–59	57·8	−·95	1·0	−1·12	−·17
Greenwich	1857–58	57·9	−·75	3·5	−1·13	−·38
Königsberg	1857–59	58·0	−1·55	1·0	−1·14	−·41
Paris	1858–59	59·0	−1·61	1·5	−1·21	−·40
Cape	1856–61	59·1	−·76	4·0	−1·22	+·46
Greenwich	1859–60	59·9	−·96	3·0	−1·27	+·31
Paris	1860–62	61·6	−1·79	1·5	−1·35	−·44
Melbourne	1860–63	62·2	−1·56	1·0	−1·37	−·19
Greenwich	1861–63	62·6	−1·00	2·5	−1·38	+·38
Radcliffe	1862–64	62·6	−2·54	1·0	−1·38	−1·16
Cape	1861–64	62·6	−1·68	3·0	−1·38	−·30
Paris	1863–64	63·8	−2·01	1·5	−1·40	−·61
Washington	1863–65	64·3	−·92	0·5	−1·40	+·48
Cape	1865–70	64·5	−1·44	3·0	−1·40	−·04
Brussels	1857–78	64·7	−·95	4·0	−1·39	+·44
Pulkova	1861–75	64·8	−1·51	4·0	−1·39	−·12
Greenwich	1864–65	65·0	−1·04	3·5	−1·39	+·35
Paris	1865–66	65·5	−1·83	1·0	−1·38	−·45
Radcliffe	1865–67	66·5	−1·31	1·0	−1·36	+·05
Leiden	1864–70	66·8	−1·65	2·0	−1·35	−·30

TABLE II.—*continued.*

Observatory.	Period of Observation.	Mean Epoch.	Observed *r*.	Weight.	Computed *r*.	O−C.
			″		″	″
Greenwich	1866–67	1866·9	−1·38	3·0	−1·35	− ·03
Washington	1866–67	66·9	−1·10	2·0	−1·35	+ ·25
,,	1867–68	67·4	−1·33	4·5	−1·33	·00
Melbourne	1863–70	67·9	−1·24	2·5	−1·31	+ ·07
Paris	1867–68	67·9	−1·55	1·5	−1·31	− ·24
Washington	1868–69	68·7	−1·07	2·0	−1·27	+ ·20
Radcliffe	1868–69	69·4	−1·94	1·0	−1·22	− ·72
Paris	1869–70	70·1	−1·36	2·0	−1·17	− ·19
Glasgow	1860–81	70·2	− ·29	2·0	−1·16	+ ·87
Radcliffe	1870	70·5	−1·10	2·5	−1·14	+ ·04
Madras	1862–87	70·7	−1·15	3·0	−1·13	− ·02
Washington	1869–72	71·0	−1·09	4·0	−1·11	+ ·02
Radcliffe	1871–72	71·6	−1·67	2·0	−1·06	− ·61
Greenwich	1868–76	72·1	− ·59	6·5	−1·01	+ ·42
Harvard	1871–72	72·2	− ·04	1·0	−1·00	+ ·96
Paris	1871–74	72·7	−1·09	3·0	− ·94	− ·15
Washington	1873–75	74·3	−1·23	4·0	− ·77	− ·46
Harvard	1870–79	74·7	− ·30	4·0	− ·72	+ ·42
Radcliffe	1873–76	75·2	− ·80	2·0	− ·65	− ·15
Paris	1875–78	76·3	−1·08	3·0	− ·53	− ·55
Melbourne	1871–85	76·6	−1·06	2·5	− ·48	− ·58
Pulkova	1874–80	76·7	− ·29	9·0	− ·47	+ ·18
Cape	1871–79	77·7	− ·58	1·0	− ·34	− ·24
Washington	1876–80	77·8	− ·86	4·0	− ·33	− ·53
Sydney	1877–81	78·2	−1·55	0·5	− ·27	−1·28
Radcliffe	1880–93	80·9	+ ·13	1·5	+ ·08	+ ·05
Greenwich	1877–86	81·3	+ ·44	6·0	+ ·13	+ ·31
Paris	1881–82	81·9	− ·52	3·0	+ ·20	− ·72
Washington	1881–83	82·4	− ·78	3·0	+ ·26	−1·04
Cape	1879–85	82·9	+ ·06	6·0	+ ·32	− ·26
Paris	1883–84	83·9	− ·23	5·0	+ ·38	− ·61
Pulkova	1882–91	84·2	+ ·23	5·0	+ ·39	− ·16
Washington	1884–88	85·6	− ·33	4·0	+ ·40	− ·73
Strasbourg	1884–88	85·7	+ ·74	8·0	+ ·40	+ ·34
Paris	1885–87	86·3	− ·06	3·5	+ ·38	− ·44
Cape	1886–87	87·0	+ ·70	5·0	+ ·33	+ ·37
Melbourne	1885–94	88·3	− ·03	1·5	+ ·19	− ·22
Glasgow	1886–92	88·3	+ ·40	1·0	+ ·19	+ ·21
Washburn	1888–90	89·2	− ·25	1·0	+ ·06	− ·31

TABLE II.—*continued.*

Observatory.	Period of Observation.	Mean Epoch.	Observed *r*.	Weight.	Computed *r*.	O−C.
			″		″	″
Cape	1888–90	1889·7	+ ·46	5·0	− ·02	+ ·48
Pulkova	1881–94	89·9	+ ·29	8·0	− ·05	+ ·34
Paris	1888–91	90·1	− ·53	1·5	− ·08	− ·45
Strasbourg	1888–93	90·7	− ·31	3·0	− ·18	− ·13
Greenwich	1887–92	91·4	+ ·41	3·5	− ·30	+ ·71
Cape	1891–94	92·3	+ ·20	4·0	− ·45	+ ·65
Nice	1888–96	93·0	− ·91	3·0	− ·56	− ·35
Paris	1892–93	93·1	−1·02	2·5	− ·57	− ·45
Greenwich	1893	93·5	− ·18	3·0	− ·63	+ ·45
Berlin	1892–97	94·1	− ·63	0·5	− ·74	+ ·11
Greenwich	1894	94·5	− ·47	3·0	− ·79	+ ·32
,,	1895	95·5	− ·49	3·0	− ·93	+ ·44
,,	1896	96·4	−1·01	2·5	−1·03	+ ·02
Cape	1896–99	97·4	−1·15	3·0	−1·14	− ·01
Greenwich	1897–98	97·9	− ·88	4·0	−1·18	+ ·30
Paris	1897–98	98·2	−1·46	2·5	−1·21	− ·25
Nice	1897–00	98·9	−1·69	2·5	−1·28	− ·41
Radcliffe	1894–03	99·5	−1·57	1·0	−1·32	− ·25
Paris	1899–00	99·6	−1·67	4·0	−1·33	− ·34
Greenwich	1899–00	99·9	−1·08	5·0	−1·35	+ ·27
Odessa	1901–02	1901·7	−1·44	2·0	−1·43	− ·01
Paris	1901–02	01·8	−1·37	4·0	−1·44	+ ·07
Greenwich	1901–02	01·9	−1·12	4·5	−1·44	+ ·32
,,	1899–08	02·1	−1·20	2·0	−1·45	+ ·25
Washington	1898–07	03·9	− ·44	1·0	−1·48	+1·04
Greenwich	1903–05	04·1	−1·11	5·0	−1·48	+ ·37
Pulkova	1903–06	05·2	−2·03	5·0	−1·48	− ·55
Paris	1903–06	05·4	−1·70	3·0	−1·48	− ·22
Cape	1905–11	07·0	−1·40	4·0	−1·43	+ ·03
Washington	1903–11	07·2	−1·37	1·0	−1·42	+ ·05
Greenwich	1906–09	07·9	−1·42	5·0	−1·38	− ·04
Lick	1907–08	08·0	−1·50	1·0	−1·37	− ·13
Greenwich	1909–13	10·2	−1·00	1·0	−1·27	+ ·27
Abbadia	1900–14	11·0	−1·26	4·0	−1·20	− ·06
Paris	1909–14	11·7	−1·25	5·0	−1·15	− ·10
Greenwich	1910–13	11·8	−1·09	5·0	−1·14	+ ·05
Cape	1912–16	12·8	−1·16	2·5	−1·05	− ·11
Pulkova	1911–18	14·1	−1·13	2·0	− ·91	− ·22
Greenwich	1915–21	17·8	− ·50	9·0	− ·45	− ·05
Abbadia	1919–21	21·0	− ·60	2·0	·00	− ·60
Cape	1918–25	22·0	− ·10	2·5	+ ·11	− ·21

The following preliminary elements were adopted, based upon graphical construction and approximate solutions :—

T	=	1885·76
P	=	40·3 years
n	=	8°·9330
i	=	+30°·6
ω	=	54°·5
Ω	=	311·0
e	=	0·29
a	=	0″·97

} Preliminary Elements.

The radial velocities, discussed later, were used to assist in fixing the values of i and ω.

The run of the residuals from these elements indicated the necessity of a correction to the proper-motions. The following expressions were therefore used as approximations to the values of q and r :—

$$q = -0^{s}\cdot0084 - 0^{s}\cdot000219(t-1850) + 0^{s}\cdot0014\cos u + 0^{s}\cdot0587\sin u,$$
$$r = -0''\cdot447 - 0''\cdot00218\,(t-1850) + 0''\cdot883\cos u - 0''\cdot145\sin u.$$

The observations were grouped into normal places, as shown in Tables III. and IV., which give in the first column the mean date, in the second column the weight, in the third column the mean observed value of q or r respectively, in the fourth column the computed value, and in the fifth column the residual (O−C).

TABLE III.

Normal Places in Right Ascension.

Date.	Weight.	Observed q.	Computed q.	O−C.
		s	s	s
1757·2	4·0	−0·048	−0·037	−0·011
67·2	1·0	+ ·060	+ ·037	+ ·023
70·6	1·0	+ ·072	+ ·063	+ ·009
75·9	0·5	+ ·029	+ ·061	− ·032
82·4	0·5	− ·022	+ ·024	− ·046
85·9	0·5	− ·031	− ·002	− ·029
95·9	0·5	− ·087	− ·055	− ·032
99·4	0·5	− ·080	− ·051	− ·029
1805·4	1·0	− ·011	+ ·006	− ·017
08·9	1·0	+ ·050	+ ·043	+ ·007
15·0	3·0	+ ·068	+ ·056	+ ·012
18·3	3·5	+ ·032	+ ·043	− ·011
22·0	6·0	+ ·009	+ ·020	− ·011
24·5	4·0	+ ·010	+ ·002	+ ·008
28·6	12·5	− ·029	− ·027	− ·002
32·1	11·0	− ·059	− ·049	− ·010
33·3	9·5	− ·052	− ·055	+ ·003

TABLE III.—*continued.*

Date.	Weight.	Observed q. s	Computed q. s	O−C. s
1835·6	7·0	−0·062	−0·062	0·000
38·3	10·0	− ·081	− ·063	− ·018
41·2	14·0	− ·054	− ·051	− ·003
43·1	12·5	− ·047	− ·035	− ·012
45·4	21·5	+ ·007	− ·007	+ ·014
46·0	7·5	− ·016	·000	− ·016
47·8	20·0	+ ·018	+ ·021	− ·003
49·7	6·5	+ ·026	+ ·038	− ·012
51·8	12·0	+ ·057	+ ·047	+ ·010
54·7	7·5	+ ·048	+ ·049	− ·001
56·0	9·0	+ ·033	+ ·045	− ·012
57·7	9·0	+ ·047	+ ·038	+ ·009
59·2	10·5	+ ·023	+ ·031	− ·008
62·0	10·5	+ ·014	+ ·013	+ ·001
63·3	13·0	+ ·007	+ ·004	+ ·003
65·3	21·0	+ ·014	− ·011	+ ·025
66·4	11·5	− ·006	− ·019	+ ·013
67·2	13·0	− ·020	− ·024	+ ·004
68·3	15·5	− ·028	− ·032	+ ·004
70·0	12·0	− ·046	− ·044	− ·002
71·9	15·0	− ·061	− ·055	− ·006
72·6	8·0	− ·062	− ·059	− ·003
74·6	12·0	− ·065	− ·068	+ ·003
76·4	12·5	− ·075	− ·072	− ·003
78·4	5·0	− ·061	− ·073	+ ·012
81·8	18·0	− ·059	− ·058	− ·001
84·1	16·0	− ·030	− ·036	+ ·006
85·8	19·0	− ·013	− ·015	+ ·002
88·9	23·0	+ ·021	+ ·019	+ ·002
92·0	16·5	+ ·054	+ ·038	+ ·016
93·7	11·0	+ ·046	+ ·041	+ ·005
97·4	22·0	+ ·034	+ ·032	+ ·002
99·2	15·0	+ ·031	+ ·023	+ ·008
1901·4	15·0	+ ·010	+ ·010	·000
03·8	9·0	− ·009	− ·007	− ·002
05·1	14·5	− ·023	− ·016	− ·007
07·1	15·0	− ·033	− ·030	− ·003
11·2	16·5	− ·067	− ·057	− ·010
13·7	17·0	− ·082	− ·072	− ·010
18·2	12·0	− ·087	− ·082	− ·005
21·5	4·0	− ·084	− ·071	− ·013

TABLE IV.

Normal Places in Declination.

Date.	Weight.	Observed q.	Computed q.	O−C.
		″	″	″
1756·6	1·0	+0·19	−0·11	+0·30
1813·6	1·5	−1·00	− ·54	− ·46
16·2	3·0	− ·71	− ·86	+ ·15
20·7	2·0	−1·54	−1·20	− ·34
24·4	7·5	−1·46	−1·29	− ·17
27·0	6·0	−1·20	−1·23	+ ·03
29·6	7·5	−1·19	−1·10	− ·09
32·3	8·5	−1·03	− ·87	− ·16
34·3	8·0	−1·00	− ·66	− ·34
38·1	12·5	− ·19	− ·16	− ·03
42·7	12·0	+ ·49	+ ·38	+ ·11
45·3	13·5	+ ·55	+ ·45	+ ·10
49·1	10·5	+ ·46	+ ·10	+ ·36
52·5	5·5	− ·49	− ·42	− ·07
55·7	8·0	− ·68	− ·86	+ ·18
57·9	5·5	− ·93	−1·08	+ ·15
59·4	8·5	− ·98	−1·20	+ ·22
62·4	9·0	−1·59	−1·35	− ·24
64·5	13·0	−1·36	−1·37	+ ·01
66·2	12·5	−1·31	−1·35	+ ·04
67·8	10·5	−1·29	−1·30	+ ·01
70·3	10·5	−1·09	−1·25	+ ·16
71·9	16·5	− ·90	−1·03	+ ·13
74·6	10·0	− ·77	− ·74	− ·03
76·6	14·5	− ·59	− ·49	− ·10
77·8	5·5	− ·87	− ·33	− ·54
81·7	13·5	− ·08	+ ·17	− ·25
83·4	11·0	− ·07	+ ·32	− ·39
85·2	17·0	+ ·34	+ ·37	− ·03
87·1	11·0	+ ·33	+ ·28	+ ·05
89·8	15·5	+ ·23	− ·05	+ ·28
91·5	10·5	+ ·12	− ·31	+ ·43
93·6	12·0	− ·63	− ·63	·00
96·9	12·5	− ·88	−1·04	+ ·16
99·3	15·0	−1·44	−1·26	− ·18
1901·9	12·5	−1·26	−1·41	+ ·15
04·8	14·0	−1·52	−1·46	− ·06
07·5	11·0	−1·41	−1·41	·00
11·4	15·0	−1·18	−1·18	·00
13·4	4·5	−1·15	−1·00	− ·15
17·8	9·0	− ·50	− ·46	− ·04
21·6	4·5	− ·32	+ ·04	− ·36

Normal equations were formed from the equations of condition derived from these normal places, as explained above. It was found that there was a strong correlation between the coefficients of G and τ in right ascension and between the coefficients of H′ and τ in declination. It was therefore considered preferable not to solve for τ separately, but to solve for the other unknowns in terms of τ. The normal equations in right ascension were

A.	B.	G.	H.	E.	N.		
+582·34	+236·21	−107·20	− 29·19	+ 4·27	−17·67	=− 3·79	− 16·65τ
	+299·44	− 84·03	+ 2·93	− 6·05	− 8·95	=+ 2·07	− 47·30τ
		+342·65	+ 13·04	− 1·40	−48·79	=+ 5·84	+278·16τ
			+256·92	−11·43	− 2·60	=+16·26	− 0·34τ
				+63·74	+ 1·75	=+ 0·19	+ 2·67τ
					+27·93	=+ 0·19	− 43·14τ

the solution of which gives

A	= − 0·005 + 0·138τ	weight 360·8
B	= + 0·020 − 0·026τ	,, 193·4
G	= + 0·027 + 0·852τ	,, 213·9
H	= + 0·062 − 0·026τ	,, 251·3
E	= + 0·015 + 0·044τ	,, 62·9
N	= + 0·063 + 0·018τ	,, 18·6.

The normal equations in declination were

A′.	B′.	G′.	H′.	E.	N.		
+411·68	+194·59	− 82·20	− 25·72	−162·58	− 8·58	=+ 7·02	+ 19·05τ
	+209·26	− 40·39	+ 1·01	− 78·54	+ 8·21	=+ 5·06	+ 1·58τ
		+225·96	+ 7·24	+ 8·36	+ 5·35	=+10·14	− 30·33τ
			+185·68	+ 13·41	+19·20	=+20·84	−143·01τ
				+106·48	+ 4·47	=− 2·99	− 5·96τ
					+12·98	=+ 1·70	− 14·66τ

the solution of which gives

A′	= + 0·039 − 0·022τ	weight 114·8
B′	= + 0·008 + 0·020τ	,, 104·4
G′	= + 0·057 − 0·116τ	,, 192·4
H′	= + 0·120 − 0·774τ	,, 155·9
E	= + 0·020 + 0·031τ	,, 39·2
N	= − 0·057 + 0·025τ	,, 9·8.

For a combined solution it is necessary that the probable error corresponding to unit weight should have the same value for right ascensions and declinations. Investigation showed that the probable error is larger for the declinations, the values being $\pm$ ″·30 for right ascension and $\pm$ ″·33 for declination. The declination equations were therefore multiplied throughout by 0·826 and combined with the right

ascension equations. The solution of the combined normals then gave :

$$\begin{aligned}
A &= -0{\cdot}006 + 0{\cdot}138\tau \quad \text{weight } 360{\cdot}6 \\
B &= +0{\cdot}018 - 0{\cdot}026\tau \quad \text{,,} \quad 202{\cdot}8 \\
G &= +0{\cdot}021 + 0{\cdot}853\tau \quad \text{,,} \quad 249{\cdot}1 \\
H &= +0{\cdot}062 - 0{\cdot}027\tau \quad \text{,,} \quad 255{\cdot}3 \\
A' &= +0{\cdot}042 - 0{\cdot}018\tau \quad \text{,,} \quad 136{\cdot}1 \\
B' &= \quad 0{\cdot}000 + 0{\cdot}021\tau \quad \text{,,} \quad 93{\cdot}9 \\
G' &= +0{\cdot}055 - 0{\cdot}114\tau \quad \text{,,} \quad 172{\cdot}0 \\
H' &= +0{\cdot}112 - 0{\cdot}773\tau \quad \text{,,} \quad 144{\cdot}6 \\
E &= +0{\cdot}016 + 0{\cdot}040\tau \quad \text{,,} \quad 95{\cdot}31 \\
N &= +0{\cdot}027 + 0{\cdot}020\tau \quad \text{,,} \quad 26{\cdot}7.
\end{aligned}$$

With these values of the coefficients the sum of the squares of the residuals becomes :

$$\begin{aligned}
&\text{From R.A. equations,} \quad + 7{\cdot}49 + 14{\cdot}81\tau + 236{\cdot}41\tau^2. \\
&\text{,, Dec. ,,} \quad + 10{\cdot}82 + 21{\cdot}61\tau + 161{\cdot}67\tau^2.
\end{aligned}$$

The minimum value of the sum of the squares is obtained for $\tau = -{\cdot}0313$ in right ascension and for $\tau = -{\cdot}0669$ in declination. Combining the two series, the minimum value is obtained for $\tau = -{\cdot}045$, and for this value the probable error corresponding to unit weight is found to be $\pm 0''{\cdot}30$. This value of τ corresponds to a correction dT to the assumed time of periastron of amount $-0{\cdot}26$ years. For the present, however, it is preferable not to adopt a definite value for dT.

The corrections to the eccentricity given by the right ascensions and declinations separately are in excellent agreement, and the eccentricity may be regarded as well determined. The corrections to the mean motion (or to the period), which depend upon the quantity N, are discordant and have relatively small weight. This was to be anticipated in view of the fact that the observations only extend over three or four periods.

The assumed values of the constants and the corrected values are as follows :—

	Assumed.	*Correction.*		*Corrected.*	
a	$-0^s{\cdot}0084$	$-0{\cdot}0004$	$+0{\cdot}0092\tau$	$-0^s{\cdot}0088$	$+0{\cdot}0016d$T
b	$-0^s{\cdot}000219$	$+0{\cdot}000024$	$-0{\cdot}000035\tau$	$-0^s{\cdot}000195$	$-0{\cdot}000006d$T
g	$+0^s{\cdot}0014$	$+0{\cdot}0014$	$+0{\cdot}0569\tau$	$+0^s{\cdot}0028$	$+0{\cdot}0099d$T
h	$+0^s{\cdot}0587$	$+0{\cdot}0041$	$-0{\cdot}0018\tau$	$+0^s{\cdot}0628$	$-0{\cdot}0003d$T
a'	$-0''{\cdot}447$	$+0{\cdot}042$	$-0{\cdot}018\tau$	$-0''{\cdot}405$	$-0{\cdot}003d$T
b'	$-0''{\cdot}00218$	·000	$+0{\cdot}00040\tau$	$-0''{\cdot}00218$	$+0{\cdot}00007d$T
g'	$+0''{\cdot}883$	$+0{\cdot}055$	$-0{\cdot}114\tau$	$+0''{\cdot}938$	$-0{\cdot}020d$T
h'	$-0''{\cdot}145$	$+0{\cdot}112$	$-0{\cdot}773\tau$	$-0''{\cdot}033$	$-0{\cdot}135d$T
e	0·29	$+0{\cdot}016$	$+0{\cdot}040\tau$	0·306	$+0{\cdot}007d$T
n	$8°{\cdot}9330$	$+0{\cdot}0154$	$+0{\cdot}011\tau$	$8°{\cdot}9484$	$+0{\cdot}011d$T
P	40·30 years			40·23	$-0{\cdot}049d$T years

In the above corrected values, dT is the correction in years to time of periastron. The values of q and r as derived from observation are therefore :

$$q = -0^s\cdot0088 + 0^s\cdot0016d\mathrm{T} - (0^s\cdot000195 + 0^s\cdot000006d\mathrm{T})(t - 1850) + (0^s\cdot0028 + 0^s\cdot0099d\mathrm{T})\cos u + (0^s\cdot0628 - 0^s\cdot0003d\mathrm{T})\sin u,$$

$$r = -0''\cdot405 - 0''\cdot003d\mathrm{T} - (0''\cdot00218 - 0''\cdot00007d\mathrm{T})(t - 1850) + (0''\cdot938 - 0''\cdot020d\mathrm{T})\cos u - (0''\cdot033 + 0''\cdot135d\mathrm{T})\sin u,$$

dT being the correction to the assumed time of periastron, 1885·76.

The quantities P, Q, R, S defined above are given by

$$\mathrm{P} = +\cdot938 - \cdot020d\mathrm{T},$$
$$\mathrm{Q} = +\cdot035 + \cdot142d\mathrm{T},$$
$$\mathrm{R} = +\cdot042 + \cdot149d\mathrm{T},$$
$$\mathrm{S} = -\cdot987 + \cdot005d\mathrm{T}.$$

It will be noticed that (Q − R) and (P + S) are both small quantities, and that therefore tan $(\omega - \Omega)$, which is given by their ratio, is determined with very small weight. Also $\cos i = \phi - (\phi^2 - 1)^{\frac{1}{2}}$, where $\phi = (\mathrm{P}^2 + \mathrm{Q}^2 + \mathrm{R}^2 + \mathrm{S}^2)/2(\mathrm{QR} - \mathrm{PS})$ and is a quantity not much greater than 1 ; the value of i is therefore also weakly determined. The meridian observations are not suitable for the accurate determination of the quantities i and $(\omega - \Omega)$. On the other hand $(\omega + \Omega)$ is determined with considerable accuracy.

The radial velocity observations lead to satisfactory determinations of the quantities i and ω ; when ω has been determined, Ω can be derived from the meridian observations. The radial observations of Procyon obtained at the Cape will therefore now be considered.

II. *The Radial Observations.*

The amplitude of the radial velocity variation of Procyon is small. To provide information of value, therefore, observations with a spectroscope of high dispersion are necessary, and in order to reduce accidental errors to a minimum the spectra should be measured differentially against the same standard. The four-prism spectrograph with long camera attached to the Victoria telescope of the Cape Observatory gives a dispersion of 8·4 A. per mm. at H_γ, which is sufficiently large to give a high degree of accuracy in the measurement of a spectrum with sharply defined lines.

Forty-seven plates of Procyon were obtained in the years 1909 to 1912, and 72 plates in the years 1922 to 1924. These have all been measured differentially in the Hartmann spectrocomparator against the same standard plate, No. 2147, also of Procyon. Any error in the assumed shift of the standard plate does not enter into the derived value of the semiaxis of the orbit.

The particulars of the radial-velocity determinations are given in the following Table V.

TABLE V.

Radial Velocities of Procyon.

Epoch.	Observed Velocity km./sec.	No. of Plates.	Computed Velocity km./sec.	O−C. km./sec.
1909·35	−3·78	13	−3·84	+·06
10·63	−3·76	9	−3·70	−·06
11·45	−3·58	12	−3·61	+·03
12·44	−3·54	13	−3·49	−·05
22·10	−2·25	24	−2·25	·00
23·14	−2·25	24	−2·26	+·01
24·19	−2·37	24	−2·36	−·01

The radial velocity at any instant may be expressed in the form

$$V = G + K \cos (v + \omega),$$

where v denotes the true anomaly. K is the semi-amplitude of the velocity variation. The quantity G determines the velocity of the centre of mass of the system by the relation

$$G = V_0 + Ke \cos \omega.$$

If the radial velocity determinations extended over a whole period they would suffice to determine the eccentricity and the time of periastron passage. The velocity determinations which are available not being sufficient for this purpose it is necessary to assume certain data, viz. the period, the time of periastron passage, and the eccentricity. We assume provisionally

$$T = 1885{\cdot}50\,; \qquad P = 40{\cdot}23 \text{ years}\,; \qquad e = {\cdot}304.$$

Then the phase and mean anomaly are determined for each observation and the value of the true anomaly derived from Schlesinger's tables.* Equations of condition in the form

$$V = G + K_1 \cos v + K_2 \sin v$$

are derived and normal equations constructed for G, K_1, and K_2.

The normal equations so obtained are as follows :—

$$\begin{aligned} +\,119{\cdot}06G + 10{\cdot}00K_1 - 69{\cdot}80K_2 &= -\,336{\cdot}99, \\ +\,74{\cdot}63 \quad - 13{\cdot}27 \quad &= +\,33{\cdot}42, \\ +\,44{\cdot}53 \quad &= +\,188{\cdot}73, \end{aligned}$$

which give

$$\begin{aligned} G &= -\,3{\cdot}44 && \text{weight } 7{\cdot}88, \\ K_1 &= K \cos \omega = +\,{\cdot}742 && \text{,, } 58{\cdot}4, \\ K_2 &= -\,K \sin \omega = -\,{\cdot}940 && \text{,, } 2{\cdot}82. \end{aligned}$$

The weight of K_2 is small compared with that of K_1 on account of the correlation between the coefficients of the G and K_2 terms. Sin v has the same sign for all the observations. If they could have been continued for a few more years a better separation of G and K_2 would have resulted.

* *Allegheny Publications*, vol. ii., No. 17.

The probable error corresponding to unit weight (one plate) is found to be ± 0·12, and we obtain

$$\begin{aligned} G &= -3{\cdot}44 \pm 0{\cdot}042 \text{ km./sec.}, \\ K &= 1{\cdot}198 \pm 0{\cdot}057 \text{ km./sec.}, \\ \omega &= +51^\circ{\cdot}7 \pm 2^\circ{\cdot}1. \end{aligned}$$

The radial velocities computed from these elements are given in Table V. above. In no case do the residuals exceed a few hundredths of a kilometre a second.

The value found for G corresponds to the radial velocity of the centre of mass of the system

$$V_0 = -3{\cdot}67 \text{ km./sec.},$$

whilst the maximum and minimum velocities are respectively

$$-2{\cdot}25 \text{ km./sec} \qquad \text{and} \qquad -4{\cdot}64 \text{ km./sec.}$$

The small residuals in Table V. illustrate the precision of radial velocity measurements of a good spectrum photographed with a spectroscope of high dispersion and measured differentially. Owing to the temporary cessation of radial velocity observations at the Cape it is not possible to secure any further observations of Procyon for the time being. A further series of plates obtained in a few years' time would greatly strengthen the determination of K and ω.

From the value derived for K, the quantity $a \sin i$ can be deduced from the relation

$$K = \mu a \sin i (1 - e^2)^{-\frac{1}{2}},$$

where μ is the mean angular motion, or

$$a \sin i = [6{\cdot}70092]\text{K.P.}(1-e^2)^{\frac{1}{2}},$$

where K is expressed in km./sec., P in years, and a in kilometres. Thus

$$a \sin i = 2{\cdot}3061 \times 10^8 \text{ kms.}$$

The parallax of Procyon is fairly well determined. Schlesinger * gives the following determinations :—

McCormick,	$0''{\cdot}312 \pm 0''{\cdot}009$,
Yerkes,	$0''{\cdot}314 \pm 0''{\cdot}012$,
Yale,	$0''{\cdot}340 \pm 0''{\cdot}015$,
Sproul,	$0''{\cdot}278 \pm 0''{\cdot}016$.

Adopting the value $0''{\cdot}308$, the resulting value for $a \sin i$ in seconds of arc is

$$a \sin i = 0''{\cdot}475,$$

which, in conjunction with the value of a derived from the meridian observations, provides a good determination of the value of the inclination of the orbit.

* *General Catalogue of Parallaxes*, p. 48.

The results deduced above from the radial velocity observations are based upon assumed values of the period, eccentricity and time of periastron passage. The values of the period and eccentricity obtained from the discussion of the meridian observations are also dependent to a small extent upon the time of periastron passage. Small changes in these two quantities have very little influence on the results deduced from the radial velocity observations ; it is otherwise, however, with the time of periastron passage, a small change in which produces a large change in ω.

To allow for the dependence of ω upon the assumed value of the time of periastron passage, the radial velocity observations were re-discussed, using the same eccentricity and period * but with 1886·0 as the time of periastron passage. The results of the two discussions are given below for comparison :—

T.	G.	K.	ω.	V_0.
1885·5	−3·44 km./sec.	1·198 km./sec.	+51°·7	−3·67 km./sec.
1886·0	−3·51 ,,	1·258 ,,	+58°·7	−3·71 ,,

The results of the discussion of the meridian observations have been given in a form involving corrections, dT, to an assumed time of periastron passage, 1885·76. It is convenient to express the results of the discussion of the radial velocity observations in the same form. If therefore we write

$$T = 1885{\cdot}76 + dT,$$

we derive

$$\begin{aligned} K &= 1{\cdot}229 && + 0{\cdot}120 dT \text{ km./sec.} \\ \omega &= +55°{\cdot}3 && + 14°{\cdot}0 dT \\ a \sin i &= 0''{\cdot}487 && + 0''{\cdot}048 dT \\ V_0 &= -3{\cdot}69 && - 0{\cdot}08 dT \text{ km./sec.} \end{aligned}$$

It may be mentioned that any error in the assumed velocity of the standard reference plate, No. 2147, will enter with full effect into the value of V_0, but will not affect the remaining quantities.

III. *Combination of Meridian and Radial Velocity Observations.*

The two preceding sections contain the results derived from the meridian and radial velocity observations. It has been thought advisable to give these in some detail ; they contain the direct results from a discussion of the observations in which any personal bias on the part of the investigator cannot enter, except in the matter of the weighting of the meridian observations. An attempt to combine the two series reveals, however, discordances which may be dealt with in a different manner by different investigators. i and ω can be determined, though with small weight, from the meridian observations alone, but the values so derived are not in agreement with those given by the

* A third solution was made, with T=1886·0, but using the corresponding eccentricity and period derived from the meridian observations. The differences in K and ω between the second and third solutions were much smaller than the probable errors of these quantities.

radial velocities. Conversely, the latter values may be used and the coefficients g, h and g', h' in the expressions for q and r can be used to determine $a \sin \Omega$ and $a \cos \Omega$ and hence a and Ω. The values of Ω which are obtained from right ascensions and declinations differ by several degrees. To obtain agreement, different values of T are required for right ascensions and declinations.

To obtain a suitable compromise and to ascertain most easily the result of varying the elements, the meridian observations were grouped into a series of normal places of approximately equal weight covering a single period. For this purpose a period of 40·23 years was definitely adopted; the effect of any reasonable change of assumed time of periastron on the period is within the probable error of the determination. Corrections on account of proper-motion as derived above were also applied as follows :—

$$\text{in } \Delta\alpha \quad -{}^{s}\cdot0080 - {}^{s}\cdot000195(t - 1850),$$
$$\text{in } \Delta\delta \quad -{}''\cdot120 - {}''\cdot00218(t - 1850).$$

The normal places so obtained are given for the right ascensions in Table VI., and for the declinations in Table VII., the phases being reckoned from T = 1885·76.

TABLE VI.

Normal Places in R.A. reduced to One Period.

Phase.	Observed $\Delta\alpha$.	Wt.	Computed $\Delta\alpha$.	O−C.	Final Phase.	Computed $\Delta\alpha$ (Final Elements).	O−C (Final Elements).
y	s		s	s	y	s	s
1·64	+·0286	31·0	+·0241	+·0045	0·90	+·0237	+·0049
3·34	+·0316	31·5	+·0428	−·0112	2·60	+·0430	−·0114
5·83	+·0647	29·0	+·0585	+·0062	5·09	+·0594	+·0053
8·75	+·0597	34·0	+·0613	−·0016	8·01	+·0620	−·0023
11·93	+·0494	32·0	+·0510	−·0016	11·19	+·0511	−·0017
13·47	+·0413	29·0	+·0429	−·0016	12·73	+·0426	−·0013
16·22	+·0244	28·5	+·0250	−·0006	15·48	+·0246	−·0002
18·08	+·0115	28·5	+·0116	−·0001	17·34	+·0101	+·0014
19·59	+·0136	32·5	+·0001	+·0135	18·85	−·0001	+·0137
21·10	−·0048	30·0	−·0110	+·0062	20·36	−·0110	+·0062
22·48	−·0180	31·5	−·0212	+·0032	21·74	−·0208	+·0028
24·44	−·0312	27·0	−·0346	+·0034	23·70	−·0336	+·0024
26·21	−·0537	27·0	−·0455	−·0082	25·47	−·0439	−·0098
27·32	−·0527	27·0	−·0514	−·0013	26·58	−·0493	−·0034
28·78	−·0562	31·0	−·0578	+·0016	28·04	−·0550	−·0012
31·71	−·0627	31·5	−·0638	+·0011	30·97	−·0604	−·0023
34·25	−·0619	29·0	−·0589	−·0030	33·51	−·0557	−·0062
36·53	−·0417	29·5	−·0438	+·0021	35·79	−·04··	+·0001
38·77	−·0192	26·5	−·0184	−·0008	38·03	−·0180	−·0012
40·15	+·0040	33·5	+·0008	+·0032	39·41	+·0003	+·0037

TABLE VII.

Normal Places in Dec. reduced to One Period.

Phase.	Observed $\Delta\delta$.	Wt.	Computed $\Delta\delta$.	O−C.	Final Phase.	Computed $\Delta\delta$. (Final Elements).	O−C (Final Elements).
y	″		″	″	y	″	″
1·25	+0·644	14·5	+0·605	+·039	0·51	+0·531	+0·113
4·06	+ ·466	19·0	+ ·295	+·171	3·32	+ ·172	+ ·294
6·32	− ·062	16·5	− ·053	−·009	5·58	− ·200	+ ·138
9·21	− ·434	17·3	− ·479	+·045	8·47	− ·635	+ ·201
11·88	− ·828	16·5	− ·802	−·026	11·14	− ·939	+ ·111
13·81	−1·059	17·3	− ·983	−·076	13·07	−1·095	+ ·036
16·54	−1·230	19·4	−1·154	−·076	15·80	−1·221	− ·009
18·70	−1·087	18·2	−1·218	+·131	17·96	−1·246	+ ·159
19·62	−1·441	14·9	−1·225	−·216	18·88	−1·237	− ·204
21·45	−1·193	16·5	−1·208	+·015	20·71	−1·188	− ·005
22·74	−1·170	16·9	−1·169	−·001	22·00	−1·128	− ·042
25·12	− ·880	17·3	−1·038	+·158	24·38	− ·966	+ ·086
26·21	− ·777	17·8	− ·954	+·177	25·47	− ·872	+ ·095
27·72	− ·909	17·3	− ·812	−·092	26·98	− ·719	− ·190
30·14	− ·597	17·3	− ·530	−·067	29·40	− ·432	− ·165
31·91	− ·298	17·8	− ·287	−·011	31·17	− ·197	− ·101
34·10	+ ·110	16·9	+ ·041	+·069	33·36	+ ·104	+ ·006
36·88	+ ·166	19·0	+ ·436	−·270	36·14	+ ·446	− ·280
38·80	+ ·401	14·5	+ ·613	−·212	38·06	+ ·586	− ·185
40·04	+ ·527	14·9	+ ·651	−·124	39·30	+ ·601	− ·074

The weights given in Table VII. have been reduced to the basis of the weights in right ascension by multiplying by 0·826. The average weight of a normal place in R.A. is 30·0 and in Dec. is 17·0. A normal place in R.A. has therefore nearly double the weight of a normal place in Dec., and a better agreement between the observed and computed places is to be expected for right ascensions than for declinations. The values of $\Delta\alpha$, $\Delta\delta$ computed from the values of g, h and g', h' which resulted from the discussion of the meridian observations are given in column 4 of the above tables. The residuals, in the sense (O−C) are given in column 5. These computed values give the best representation of $\Delta\alpha$, $\Delta\delta$ separately. Elements adjusted to give agreement with the radial velocity observations will lead to a general increase in the residuals. The last three columns of the tables contain the phases, the values of $\Delta\alpha$, $\Delta\delta$, and the residuals which result from the finally adopted elements.

The double-star observations of Procyon indicate that the zero of position angle occurred about 1905·0; in other words, at that epoch $\Delta\alpha$ was zero. In approximating between the R.A. and Dec. observations, regard was paid to this fact. It was found not to be possible

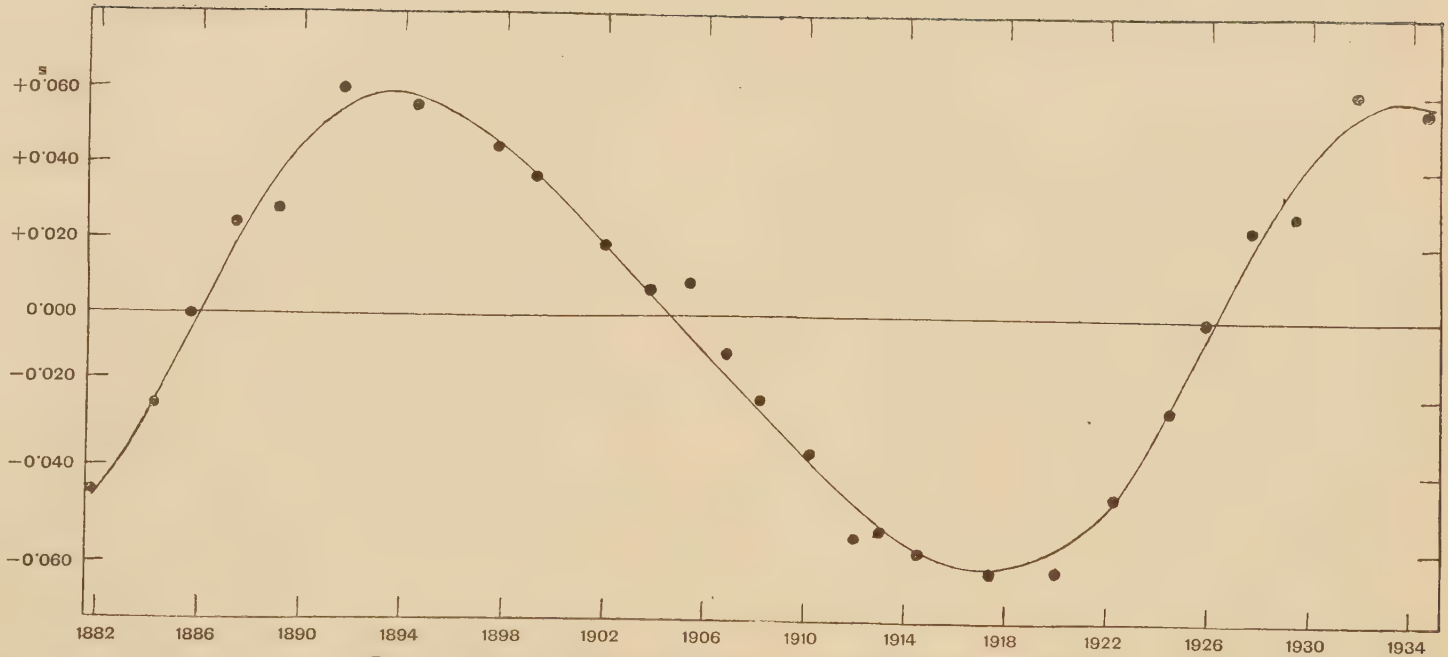


FIG. 1.—Orbital Motion in Right Ascension: Procyon (bright component).

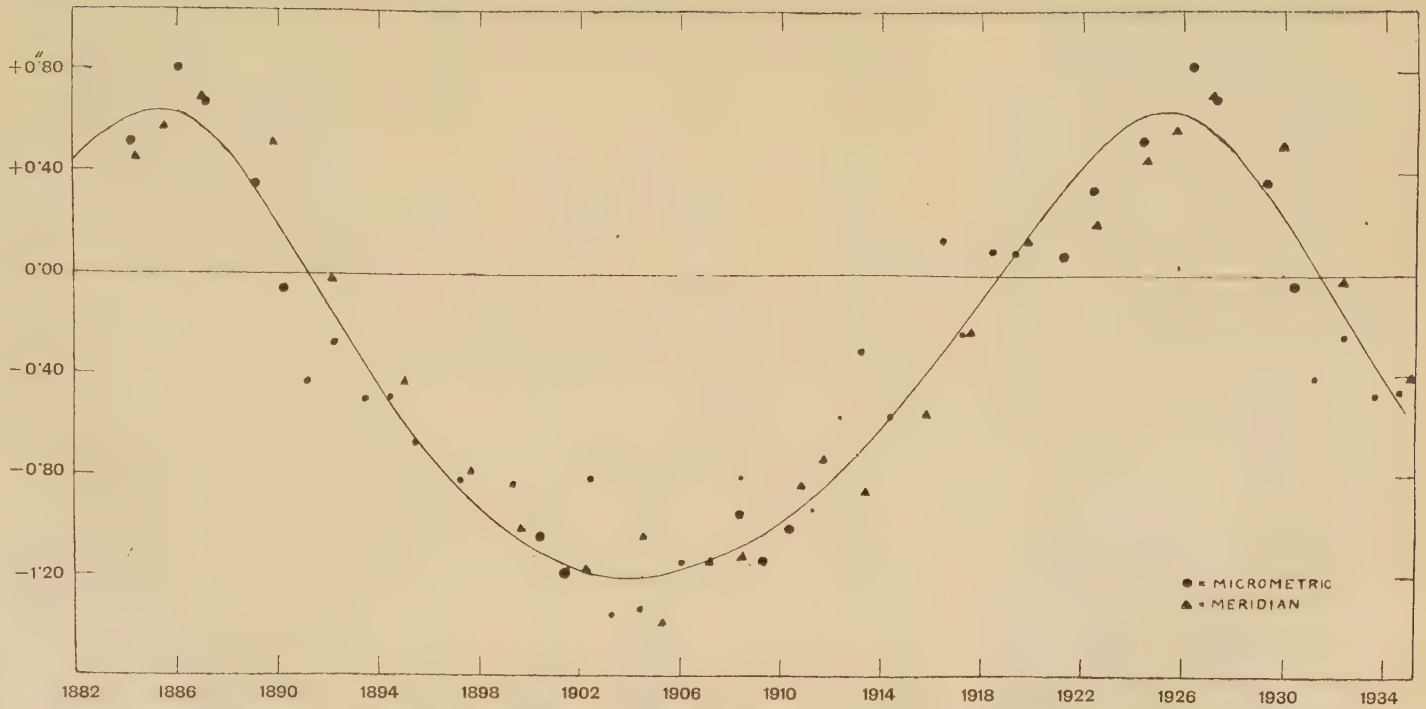


FIG. 2.—Orbital Motion in Declination: Procyon (bright component).

to secure at the same time a satisfactory representation of the right ascension and declination observations ; a greatly improved agreement would have been possible if a different time of periastron passage had been used for the two co-ordinates. The explanation may be that the observations of a bright star like Procyon are affected by systematic errors which are different in the two co-ordinates. The right ascensions give a much smoother curve than the declinations, and the elements finally adopted agree better with observation in right ascension than they do in declination. Auwers and Boss in their investigations both found that a more satisfactory representation of right ascensions than of declinations was possible.

The elements adopted finally after many trials as securing the best general agreement with the various observations are as follows :—

T	= 1886·50	Adopted Elements.
P	= 40·23 years	
n	= 8°·9485	
i	= +30°·6	
ω	= 65°·7	
Ω	= 307°·8	
e	= 0·310	
a	= 1″·020	

The radial velocity observations are satisfactorily represented by these elements, in the form

$$V = -3{\cdot}59 + 1{\cdot}320 \cos (v + 65{\cdot}7)$$

the (O−C) residuals being + ·05, − ·08, − ·02, − ·08, + ·03, + ·03, ·00.

The value 1·320 for K corresponds to

$$a \sin i = +''{\cdot}523,$$

which with $a = 1''{\cdot}020$ gives $i = +30°{\cdot}9$, in satisfactory agreement with the adopted value.

From these elements we derive

$$\Delta\alpha = -^{s}{\cdot}0033 + ^{s}{\cdot}0105 \cos u + ^{s}{\cdot}0607 \sin u,$$
$$\Delta\delta = -''{\cdot}275 + ''{\cdot}888 \cos u - ''{\cdot}260 \sin u.$$

The values of $\Delta\alpha$, $\Delta\delta$ computed by these formulæ are given in the seventh columns of Tables VI. and VII. above. Constants of $+^{s}{\cdot}0045$ in R.A. and of $-0''{\cdot}046$ in Dec. have been added to the computed values to secure the best general fit. This amounts only to a modification of the constant terms in q and r consequent upon the adoption of different orbital elements. The sixth columns of Tables VI. and VII. give the final phases and the last columns contain the (O−C) residuals.

The residuals in declination given in the last column of Table VII. appear to be rather systematic in nature. The mean residual in R.A. is $\pm\, 0^{s}{\cdot}0041$, and the mean residual in Dec. is $\pm\, 0''{\cdot}124$. These residuals

correspond to probable errors for a normal place of $\pm$ ″·052 in right ascension and of $\pm$ ″·105 in declination. These errors may be compared with the probable errors to be expected on the basis of a probable error corresponding to unit weight of $\pm$ 0″·30. The mean weights of a normal place in right ascension being 30 and in declination being 17, the probable errors to be expected are $\pm$ ″·055 in right ascension and $\pm$ ″·104 in declination, which are in close agreement with those actually found. The representation of the meridian observations by the adopted elements may therefore be considered satisfactory. Figures 1 and 2 show the curves representing the orbital motion in right ascension and declination respectively, together with the observed values for the normal places as given in the second columns of Tables VI. and VII.

IV. *The Double-Star Observations.*

The measures of the faint companion of Procyon made since the first observation by Schaeberle in 1896 are given in Table VIII. For the details of the observations later than those contained in Burnham's Double-Star Catalogue I am indebted to Prof. R. G. Aitken.

Table VIII.

Observations of the Companion of Procyon.

Date.	θ (Observed).	ρ (Observed).	No. of Nights.	Observer.	θ (Computed).	$\delta\theta$ (O−C).	ρ (Computed).	$\delta\rho$ (O−C).
	°	″			°	°	″	″
1896·93	320·4	4·63	4*n*	Schaeberle	317·9	+ 2·5	4·65	− ·02
97·00	321·1	4·83	4	A	318·1	+ 3·0	4·66	+ ·17
97·16	319·8	4·65	3	Hu	319·1	+ 0·7	4·68	− ·03
97·82	324·4	4·66	6	Schaeberle	323·0	+ 1·4	4·77	− ·11
97·83	327·4	4·80	1	See	323·1	+ 4·3	4·77	+ ·03
97·83	328·6	4·82	1	Boothroyd	323·1	+ 5·5	4·77	+ ·05
97·88	323·8	4·70	3	A	323·4	+ 0·4	4·77	− ·07
98·21	325·3	4·82	3, 2	A	325·2	+ 0·1	4·81	+ ·01
98·21	326·0	4·83	6	Barnard	325·2	+ 0·8	4·81	+ ·02
98·24	326·0	4·26	1	Lewis	325·4	+ 0·6	4·81	− ·55
98·28	325·0	4·50	1	Hu	325·6	− 0·6	4·81	− ·31
98·88	330·6	4·97	4	A	329·0	+ 1·6	4·87	+ ·10
99·07	330·6	4·91	3	Barnard	330·6	0·0	4·89	+ ·02
99·96	334·5	4·88	3	A	335·0	− 0·5	4·95	− ·07
1900·05	336·0	5·09	6	Barnard	335·5	+ 0·5	4·96	+ ·13
00·23	338·3	4·83	1	Lewis	336·5	+ 2·8	4·97	− ·14
00·29	332·4	4·60	4	See	336·7	− 4·3	4·97	− ·37
01·20	338·5	5·13	3, 2	A	341·6	− 3·1	5·01	+ ·12
01·30	337·9	5·00	5	See	342·1	− 4·2	5·02	− ·02

TABLE VIII.—*continued.*

Date.	θ (Observed).	ρ (Observed).	No. of Nights.	Observer.	θ (Computed).	$\delta\theta$ (O−C).	ρ (Computed).	$\delta\rho$ (O−C).
	°	″			°	°	″	″
1901·88	343·5	5·06	2	Barnard	345·1	− 1·6	5·03	+ ·03
02·21	344·9	5·39	2	Lewis	346·9	− 2·0	5·03	+ ·36
02·24	346·5	5·11	1	Hu	347·0	− 0·5	5·03	+ ·08
02·25	338·4	5·35	3	See	347·0	− 8·6	5·03	+ ·32
02·72	350·6	5·33	3	A	349·5	+ 1·1	5·04	+ ·29
03·15	351·0	5·16	3	Barnard	351·7	− 0·7	5·04	+ ·12
04·29	355·2	4·93	2	Bowyer	357·6	− 2·4	5·02	− ·09
04·80	357·4	5·36	1	Barnard	0·3	− 2·9	5·01	+ ·35
05·14	6·2	5·14	1	A	2·1	+ 4·1	5·00	+ ·14
05·17	5·3	4·46	1	Lewis	2·3	+ 3·0	4·99	− ·53
05·99	10·2	..	1	A	6·6	+ 3·6	..	..
06·77	28·2	..	1	A	10·8	+17·4	..	..
09·16	22·5	5·26	4, 3	Barnard	24·2	− 1·7	4·79	+ ·47
09·30	22·5	5·04	1	Bowyer	25·0	− 2·5	4·78	+ ·26
10·10	24·3	5·21	2, 1	Barnard	29·8	− 5·5	4·72	+ ·49
11·07	28·6	4·69	2	Jonckheere	35·5	− 6·9	4·64	+ ·05
13·14	34·3	..	1	Bowyer	48·7	−14·4	..	..
13·16	42·6	5·09	3	Barnard	48·9	− 6·3	4·49	+ ·60
14·30	29·0	6·14	1	Bowyer	56·8	−27·8	4·40	+ ·85
14·94	27·5	5·25	1	Jonckheere	60·9	−33·4	4·35	+ ·90
17·24	47·4	4·12	1	,,	77·8	−30·4	4·14	− ·02
18·22	58·8	4·63	1	,,	85·5	−26·7	4·05	+ ·58
21·21	94·1	5±	1	Bernewiz	112·2	−18·1	3·65	+1·3±
24·19	106±	5·5±	1	Dick	147·1	−41·±	3·07	+2·4±

Note.—Companion not seen by Barnard, 1914, 1916, nor by Aitken from 1914 onwards.

The first five columns of the table give details of the observations and are self-explanatory. Columns 6 and 8 give the computed position angle and distance of the faint companion relative to the bright star. The final elements derived above have been used and the computed distance of the bright star from the centre of gravity has been multiplied by the factor 4·18 to give the distance of the bright star from the faint star.

The general agreement between theory and observation is satisfactory from the date of discovery of the faint companion until 1913, when the difficulty of the observation is taken into consideration. The observations from 1914 onwards give such large residuals that it is difficult to believe that they represent real observations of the companion, particularly in view of the fact that such a keen-sighted observer as Barnard, observing with a large instrument and under favourable

conditions, was unable to see the companion after 1913. Aitken also has looked for the companion without success on many nights from 1914 onwards.

Burnham looked carefully for a companion in 1888 and 1890, using the 36-inch refractor of the Lick Observatory, but without success. The ephemeris gives for the distances at the dates of Burnham's observations 2″·7 and 3″·2 respectively ; the companion was then undoubtedly too close to be observed.

According to the ephemeris computed from the elements derived in the present investigation, the distance of the faint companion at the beginning of 1911 was the same as at its discovery by Schaeberle towards the end of 1896. The concordance of the observations obtained within a year of the discovery would appear to indicate that the companion was not then a particularly difficult object. The large residuals obtained from 1909 onwards show that the observation of the companion was then more difficult than at discovery ; it is not easy to account for this unless the companion is variable in brightness and suffered a marked diminution in brightness at some time after 1906. The distance between the two stars is now too small for observations to be possible, but the companion should be carefully searched for again about 1935.

In view of the many spurious announcements of the observation of a close faint companion to Procyon which had been made before the discovery of the true companion by Schaeberle, it is doubtful whether the observations from 1914 onwards are real observations of the companion. With decreasing distance and probably with decreasing brightness also, it is probable that observation of the companion was no longer possible in these years.

The double-star observations indicate a ratio for the radius vector of the relative orbit to that of the orbit of the bright star about the centre of gravity of 4·18 to 1 ; in other words, the ratio of the mass of the bright star to that of the faint star is 3·18 to 1. The value of the ratio found by Boss was 3·05 to 1.

V. *The Micrometric Observations of Adjacent Stars.*

It remains to show to what extent the adopted elements satisfy the micrometric observations of the bright star relative to adjacent stars. The most complete observations of this nature are those of Otto Struve,* who commenced in 1851 a series of measures of Procyon with respect to two neighbouring stars which was continued for a period of thirty-nine years, practically covering therefore an entire revolution, with the same instrument and the same methods of observing. The two stars chosen were B.D. 5° 1738, mag. 9 and B.D. 5° 1741, mag. 8·8, which precede and follow Procyon respectively at intervals of some 20 seconds of time, and differ from Procyon in declination only by about one minute of arc.

The declination observations alone are of suitable precision for controlling the adopted elements. Struve gives † as the probable

* *Annals of the Pulkova Observatory*, vol. x.

† *Loc. cit.*, p. 26.

error of an observation of unit weight $\pm 0''\cdot654$ in R.A., and $\pm 0''\cdot157$ in Dec., unit weight corresponding to one night's complete observations, viz. four comparisons in two positions of the micrometer. The right ascension differences were obtained by transits by the eye and ear method, and were taken only as a general check.

A few observations of the position of Procyon relative to the same two stars were secured by Auwers between the years 1863 and 1873.* These can be used to supplement the observations of Struve.

Observations of Procyon relative to other neighbouring stars have been made at various times. In particular, observations are available of the distant (visual) companion from the first extant observation by Lamont in 1836 to Barnard's last observation in 1916. (Procyon, A and C, in Burnham's General Catalogue.) These observations reveal in a general manner the orbital motion of the bright star, but they are insufficient in number and accuracy to be of value for the determination of the elements of the system.

In Table IX. are given the results of Struve's observations of the two stars, B.D. 5° 1738 and B.D. 5° 1741 (stars *b* and *c*) in the form $P - \frac{b+c}{2}$ (reduced to 1850·0). An early observation by Bessel, the observations by Auwers and observations by Quetelet at Brussels are included. The first column of the table gives the name of the observer, the second the mean date, the third the declination difference $P - \frac{b+c}{2}$, and the fourth the weight of the observation. The fifth column gives the declination differences after correction for difference at epoch and for proper-motion have been applied. These corrections were determined by a least-squares solution and are as follows :—

$$+ 70''\cdot82 + 1''\cdot01855(t - 1870).$$

The figures in this column, apart from errors of observation, represent the declination changes which result from the orbital motion. The sixth column contains the values of the orbital motion in declination computed from the elements derived above. The (O−C) residuals are contained in the last column of the table.

TABLE IX.

Observer.	Date.	$P-\frac{1}{2}(b+c)$.	Wt.	$\Delta\delta$ (Observed).	$\Delta\delta$ (Computed).	O−C.
Bessel	1822·12	−22″·72	0·10	−0″·67	−1″·18	+0·51
Struve	51·12	−52·01	2·00	−0·42	−0·03	−0·39
,,	52·04	−52·78	2·00	−0·26	−0·19	−0·07
,,	53·26	−54·24	2·00	−0·49	−0·38	−0·11
,,	54·19	−55·19	2·00	−0·48	−0·51	+0·03
,,	55·25	−56·46	2·00	−0·67	−0·65	−0·02

* *Monatsberichte d. Akad. d. Wiss. zu Berlin*, p. 397, 1873.

TABLE IX.—*continued.*

Observer.	Date.	$P-\frac{1}{2}(b+c)$.	Wt.	$\Delta\delta$ (Observed).	$\Delta\delta$ (Computed).	O−C.
Struve	1857·22	−58″·61	2·00	−0″·81	−0″·86	+0·05
,,	59·20	−60·65	2·00	−0·83	−1·04	+0·21
Quetelet	60·14	−61·07	0·10	−0·30	−1·09	+0·79
Struve	60·25	−61·93	4·00	−1·04	−1·10	+0·06
,,	61·23	−63·07	5·00	−1·19	−1·15	−0·04
,,	62·21	−63·69	2·00	−0·81	−1·18	+0·37
Auwers	63·14	−65·60	0·25	−1·77	−1·19	−0·58
Struve	63·21	−65·25	2·00	1·35	−1·19	−0·16
,,	64·23	−66·27	2·00	−1·33	−1·20	−0·13
,,	66·23	−68·13	2·00	−1·15	−1·17	+0·02
Auwers	68·21	−69·80	1·00	−0·81	−1·08	+0·27
Struve	68·25	−69·98	4·00	−0·95	−1·08	+0·13
Auwers	69·19	−71·31	0·25	−1·32	−1·03	−0·29
Struve	69·22	−71·16	4·00	−1·14	−1·03	−0·11
,,	70·22	−72·05	4·00	−1·01	−0·95	−0·06
Auwers	71·21	−72·99	0·40	−0·94	−0·87	−0·07
,,	72·20	−73·63	0·95	−0·57	−0·78	+0·21
Struve	73·22	−74·41	2·00	−0·31	−0·68	+0·37
Auwers	73·23	−75·13	0·25	−1·02	−0·67	−0·35
Struve	74·17	−75·63	2·00	−0·57	−0·57	0·00
,,	76·23	−77·03	2·00	+0·13	−0·33	+0·46
,,	77·19	−78·39	2·00	−0·25	−0·19	−0·06
,,	78·23	−79·12	2·00	+0·08	−0·04	+0·12
,,	79·18	−80·09	2·00	+0·08	+0·09	−0·01
,,	81·24	−82·21	4·00	+0·06	+0·35	−0·29
,,	82·25	−82·96	4·00	+0·33	+0·47	−0·14
,,	84·21	−84·77	4·00	+0·52	+0·63	−0·11
,,	86·21	−86·52	4·00	+0·81	+0·63	+0·18
,,	87·22	−87·68	4·00	+0·68	+0·55	+0·13
,,	89·25	−90·07	4·00	+0·35	+0·30	+0·05
,,	90·09	−91·34	4·00	−0·06	+0·17	−0·23

The agreement between observation and the derived elements is excellent. The residuals are in general small, the few large residuals being due obviously to errors of observation. There are no systematic runs in the signs of the residuals. In fact, the derived elements are in much better agreement with the declinations derived from micrometric observations than with those given by the meridian observations. Inasmuch as the elements deduced in the present investigation were

obtained without using Struve's micrometric observations, the close agreement shown in the above table affords valuable confirmation of these elements. The representation of the micrometric observations by the adopted elements is shown in figure 2, the micrometric observed values being represented by black circles whose size is proportional to the weight of the observation.

VI. *The Orbit and Mass of Procyon.*

The results of the present investigation may be summarised as follows :—

(1) The orbital elements are as follows :—

Bright Star about Centre of Gravity.	Faint Star about Brighter.
$T = 1886{\cdot}50$	$T = 1886{\cdot}50$
$P = 40{\cdot}23$ years	$P = 40{\cdot}23$ years
$n = 8^\circ{\cdot}9485$	$n = 8^\circ{\cdot}9485$
$i = +30^\circ{\cdot}6$	$i = +30^\circ{\cdot}6$
$\omega = 65^\circ{\cdot}7$	$\omega = 65^\circ{\cdot}7$
$\Omega = 307^\circ{\cdot}8$	$\Omega = 127^\circ{\cdot}8$
$e = 0{\cdot}310$	$e = 0{\cdot}310$
$a = 1''{\cdot}020$	$a = 4''{\cdot}26$

(2) The masses of the bright star and of the faint star are in the ratio 3·18 to 1.

(3) The radial velocity of the bright star is given in km./sec. by

$$V = -3{\cdot}59 + 1{\cdot}320 \cos (v + 65^\circ{\cdot}7).$$

It may be noted that the radial velocity observations require the plus sign for the inclination, the sign of which is left indeterminate by the meridian observations.

The total mass of the system can be determined, as the parallax is known, from the relationship

$$(m_1 + m_2) = (a_1 + a_2)^3/\pi^3 P^2.$$

We have

$$a_1 + a_2 = 4''{\cdot}26, \qquad \pi = ''{\cdot}308, \qquad P = 40{\cdot}23,$$

and therefore

$$(m_1 + m_2) = 1{\cdot}63 \times \odot,$$
$$m_1 = 1{\cdot}24 \times \odot,$$
$$m_2 = 0{\cdot}39 \times \odot.$$

The masses as determined by Boss are :—bright star, 1·13 ; faint star, 0·37 ; total mass, 1·50. The present investigation leads therefore to slightly larger masses than found by Boss, but the increase is not sufficient to bring Procyon into agreement with the mass-luminosity relationship as given by Eddington. The bolometric magnitude which corresponds to a mass 1·24 according to the mass-luminosity relation

is $3^{m}\cdot55$.* The observed bolometric magnitude as given by Eddington † is $2^{m}\cdot92$. The discordance between observation and the mass-luminosity relation, which was $1^{m}\cdot00$ on the basis of Boss's mass, is reduced by about one-third of its amount to $0^{m}\cdot63$. The mass to be expected for Procyon (bright component) on the basis of Eddington's mass-luminosity relation is 1·46. Can the difference between this value and the observed value of 1·24 be attributed to the uncertainties attaching to the derived elements?

If $\kappa = m_2/m_1 = a_1/a_2$ we have for the mass of the bright component

$$m_1 = \frac{{a_1}^3}{P^2 \cdot \pi^3} \cdot \frac{(1+\kappa)^2}{\kappa^3}.$$

The value of κ which has been obtained above is 1/3·18 or ·315. Substituting $a_1 = 1''\cdot02$, $\pi = ''\cdot308$, P = 40·23 we find that a value of 1·46 for m_1 corresponds to a value of ·295 for κ. With this value of κ, the radii-vectors of the relative orbit would be obtained from the corresponding radii-vectors of the orbit of the bright star about the centre of gravity by multiplying by 4·39. The observations of the faint companion given in Table VIII. will not permit a factor as large as this, which would make the residuals in the last column of the table systematically negative, with a few exceptions.

A variation of a_1 would require a corresponding change in κ in order to fit the double-star observations. If the dimensions of the double-star orbit remain unchanged we have

$$a_1 + a_2 = a_1 \cdot \frac{\kappa + 1}{\kappa} = 4''\cdot26,$$

and m_1 varies as $1/(1+\kappa)$ or is proportional to $(4''\cdot26 - a_1)$. The possible change in a_1 is only of the order of a few hundredths of a second, so that no appreciable change in m_1 can be produced in this way. It seems therefore that the mass required by the mass-luminosity relationship cannot be obtained by change of the adopted dimensions of the bright star or relative orbits; the changes which would give a mass of 1·46 would seriously violate the observations.

The only other elements on which m_1 depends are P and π. To give the required mass by change of P would necessitate a period of about 37 years, which is impossible. The uncertainty of the derived period is only about 0·5 year. A change of π from $''\cdot308$ to $''\cdot292$ would increase the mass from 1·24 to 1·46. The individual determinations of the parallax have been given above, and a value as low as $0''\cdot292$ does not seem probable, although the possibility cannot be definitely excluded. One or two additional determinations of the parallax of Procyon are to be desired.

In conclusion, it appears that no possible change in the orbital elements will give a mass for Procyon which is in accordance with

* Eddington, *Internal Constitution of the Stars*, Table 14, p. 137.
† *Ibid.*, Table 18, p. 154.

Eddington's mass-luminosity relation, but a slight reduction in the adopted parallax would reduce the discordance to a reasonable amount.

VII. *Ephemeris.*

In Table X. is given an ephemeris, covering a complete period, of :

(1) The quantities $\Delta\alpha$, $\Delta\delta$ which require to be added to the right ascension and declination of the centre of gravity of the system in order to give the position of the bright star.

(2) The position angle θ and distance ρ of the faint star relative to the bright star.

(3) The radial velocity of the bright star, V.

Table X.

Ephemeris of Procyon.

Date.	$\Delta\alpha$.	$\Delta\delta$.	θ.	ρ.	V. km./sec.
	s	"	°	"	
1908·0	−0·024	−1·10	17·7	4·86	−3·99
09·0	− ·030	−1·05	23·3	4·80	−3·87
10·0	− ·037	−0·99	29·2	4·73	−3·76
11·0	− ·043	−0·91	35·2	4·65	−3·64
12·0	− ·049	−0·82	41·4	4·58	−3·51
13·0	− ·054	−0·72	47·8	4·50	−3·39
14·0	− ·058	−0·61	54·4	4·43	−3·26
15·0	− ·061	−0·50	61·3	4·34	−3·12
16·0	− ·063	−0·37	68·5	4·25	−2·98
17·0	− ·065	−0·24	76·0	4·16	−2·84
18·0	− ·065	−0·11	83·8	4·06	−2·71
19·0	− ·063	+0·04	92·0	3·95	−2·57
20·0	− ·060	+0·17	100·7	3·82	−2·45
21·0	− ·055	+0·30	110·1	3·68	−2·35
22·0	− ·049	+0·42	120·4	3·52	−2·29
23·0	− ·040	+0·53	131·7	3·33	−2·28
24·0	− ·029	+0·60	144·5	3·12	−2·34
25·0	− ·016	+0·64	159·1	2·89	−2·52
26·0	− ·003	+0·64	176·3	2·70	−2·80
27·0	+ ·011	+0·59	195·3	2·58	−3·15
28·0	+ ·024	+0·50	215·2	2·58	−3·54
29·0	+ ·035	+0·38	234·3	2·71	−3·92
30·0	+ ·044	+0·23	251·0	2·93	−4·26
31·0	+ ·050	+0·06	265·2	3·20	−4·51
32·0	+ ·055	−0·10	277·0	3·49	−4·68
33·0	+ ·058	−0·26	287·0	3·77	−4·80
34·0	+ ·058	−0·42	295·8	4·03	−4·87
35·0	+ ·057	−0·56	303·5	4·26	4·91

TABLE X.—*continued.*

Date.	$\Delta\alpha$.	$\Delta\delta$.	θ.	ρ.	V. km./sec.
	s	″	°	″	
1936·0	+0·054	−0·69	310·3	4·46	−4·90
37·0	+ ·051	−0·81	316·7	4·63	−4·89
38·0	+ ·046	−0·91	322·8	4·76	−4·85
39·0	+ ·041	−0·99	328·5	4·86	−4·80
40·0	+ ·035	−1·06	333·9	4·94	−4·74
41·0	+ ·028	−1·12	339·3	5·00	−4·66
42·0	+ ·021	−1·16	344·6	5·02	−4·58
43·0	+ ·014	−1·19	349·8	5·04	−4·50
44·0	+ ·007	−1·20	354·9	5·03	−4·41
45·0	·000	−1·20	0·1	5·01	−4·32
46·0	− ·008	−1·18	5·4	4·97	−4·22
47·0	− ·015	−1·15	10·9	4·92	−4·11
48·0	− ·022	−1·12	16·4	4·87	−4·01
49·0	− ·029	−1·07	22·0	4·81	−3·90

Summary.

Meridian, double-star, micrometric, and radial velocity observations of Procyon have been discussed for the derivation of the orbital elements of the system and of the masses of the component stars. The derived elements are given in § VI. An ephemeris for meridian, double-star, and radial velocity observations, based on these elements, is given in § VII.

The masses of the bright and faint components are found to be 1·24 and 0·39 times the mass of the Sun respectively. The mass of the bright star is not in agreement with that given by the mass-luminosity relation. It does not seem possible to account for the discordance by the possible errors of the relevant orbital elements. A slight reduction in the adopted value of the parallax of the system would reduce the discordance.

On the Probable Error of Stellar Parallaxes. By J. J. Nassau, Ph.D.

1. It has been pointed out by Professor Eddington * that the observed distribution of a series of parallaxes may be corrected to their true distribution if the approximate value of their probable error is known. From this it is evident that when two series of parallax observations are given with different average probable errors and the stars in the series selected at random, they will have different distribution, and when corrected their true distribution will be the same.

While investigating such relations, the ultimate object being to apply corrections which will remove negative parallaxes, it was noted

* *M.N.*, **73**, 359.

that, if a series of parallax observations is divided into groups with different average probable error, the observed distribution of each group was practically the same.

This fact led to the present problem, that is, to show that the individual probable error given for each parallax determination in a series of parallax observations does not measure its precision, and that the magnitude of the probable errors in the series are distributed according to the Gaussian law irrespective of the magnitude of their parallaxes.

2. Seven hundred stellar parallaxes were used, all taken from the *Publications of the Leander McCormick Observatory*, vols. iii. and iv. They were divided into three groups as follows :

Group I. 214 parallaxes with probable errors, ± ″·005, ″·006, ″·007, and ″·008 ; average probable error, ± ″·0072.

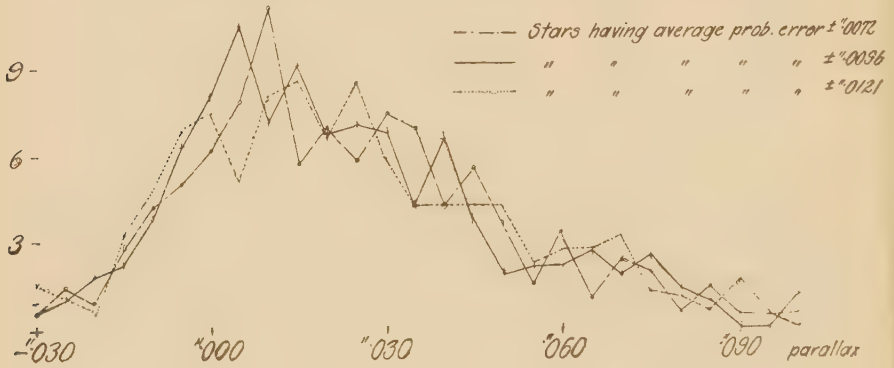


Fig. 1.

Group II. 242 parallaxes with probable errors, ± ″·009 and ″·010 ; average probable error, ± ″·0096.

Group III. 189 parallaxes with probable errors, ± ″·011, ″·012, ″·013, ″·014, ″·015, and ″·016 ; average probable error, ± ″·0121.

Fifty-two stars with parallaxes greater than ″·1 were not included, also six stars with probable errors, ± ″·004, ± ″·017, ± ″·018, or ± ″·019.

Each group was divided into consecutive intervals of ″·005 and the parallaxes in each interval grouped. The results are given in fig. 1, which shows that the observed distribution of the three groups is approximately the same.

Applying to each group the correction expressed by *

$$\bar{\epsilon} = -\frac{1}{h^2}\frac{v'(x)}{v(x)} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $\bar{\epsilon}$ is the mean error, $v(x)$ the measured or observed distribution, and $v'(x)$ its first derivative, the true distribution of each group is obtained.

* Dyson "A Method of Correcting a Series of Parallax Observations," *M.N.*, **86**, 687.

Corresponding to an interval of parallax of ″·005, the actual corrections applied are *

$$″·023\frac{\Delta y}{y} \text{ for Group I.,}$$

$$″·041\frac{\Delta y}{y} \text{ ,, ,, II.,}$$

$$″·064\frac{\Delta y}{y} \text{ ,, ,, III.,}$$

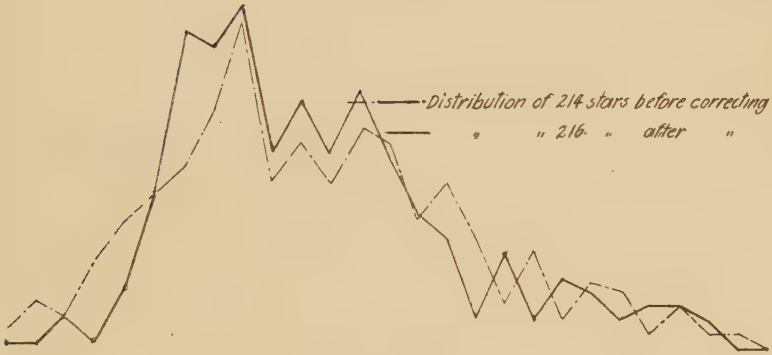


FIG. 2A.

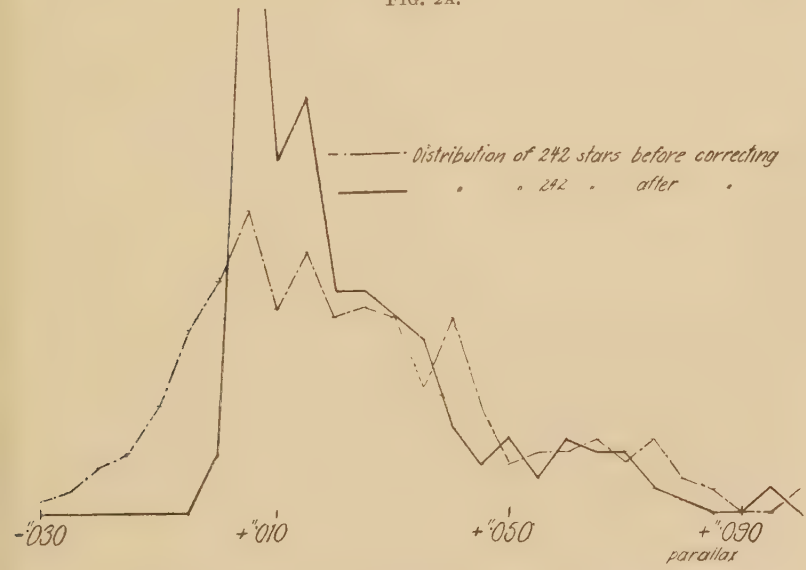


FIG. 2B.

* *M.N.*, **86**, 688.

where $y = v(x)$ and Δy the mean increment of y for an increment of x, $\Delta x = ''{\cdot}005$.

The results are given in figs. 2A, 2B, and 2C, with the apparent true distribution shown with heavy lines. If the assumed probable errors were correct and the observed distributions not all the same, the true distribution of all three groups should have been the same.

3. The curve $y = v(x)$ of the observed distribution at the negative end is of the form $y = Ae^{-k^2(x-a)^2}$, and in the case of Group III. is

$$y = 7{\cdot}4e^{-1800(x-{\cdot}01)^2}.$$

FIG. 2C.

(This is only approximate, as in this case the observations do not indicate a very definite curve.)

The correction, as expressed by formula (1), for the negative side takes the form $\epsilon = \frac{k^2}{h^2}(x-a)$. This when applied to a comparatively large negative parallax must yield a parallax less or equal to the corrected parallax of a correspondingly smaller negative parallax. That is, we must have

$$\frac{k^2}{h^2}(x+\epsilon-a) - \frac{k^2}{h^2}(x-a) \leqslant \epsilon,$$

where ϵ is positive and x and $x+\epsilon$ are two negative parallaxes. Hence $k<h$.

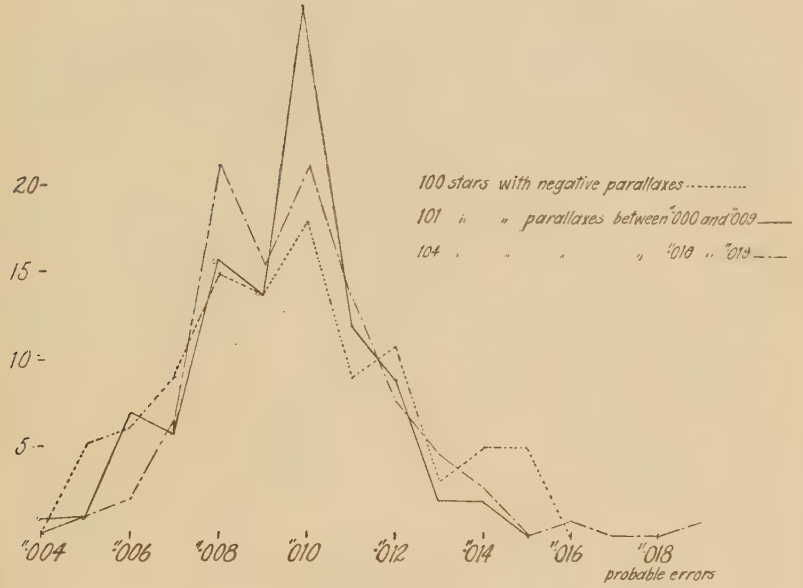

20
15
10
5
100 stars with negative parallaxes
101 " " parallaxes between ".000 and ".009
104 " " " ".018 " ".019
".004
".006
".008
".010
".012
".014
".016
".018
probable errors

FIG. 3A.

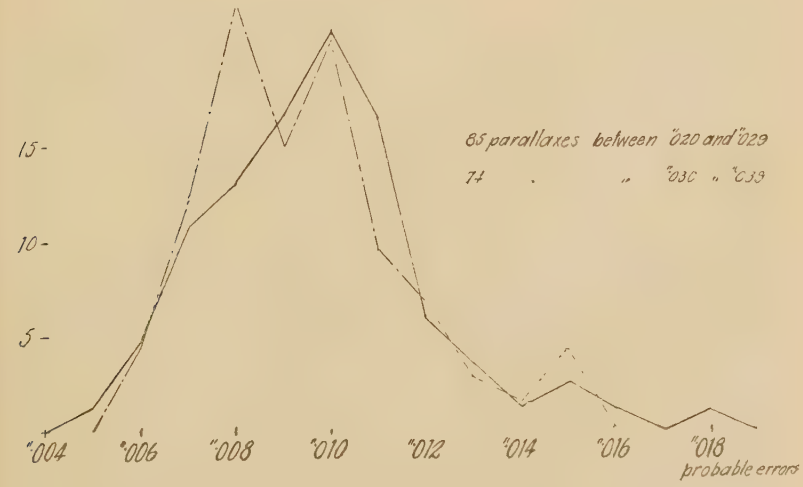

15
10
5
85 parallaxes between ".020 and ".029
74 " " ".030 " ".039
".004
".006
".008
".010
".012
".014
".016
".018
probable errors

FIG. 3B.

However, in the case of Group III. k^2 is approximately equal to 1800, $h = \frac{\cdot 477}{\cdot 0121} = 39{\cdot}5$, and hence $k > h$. Therefore the indications are that the assumed average probable error of the group is not correct.

4. An examination of the distribution of the probable errors shows that the magnitude of the parallaxes does not affect in the least their

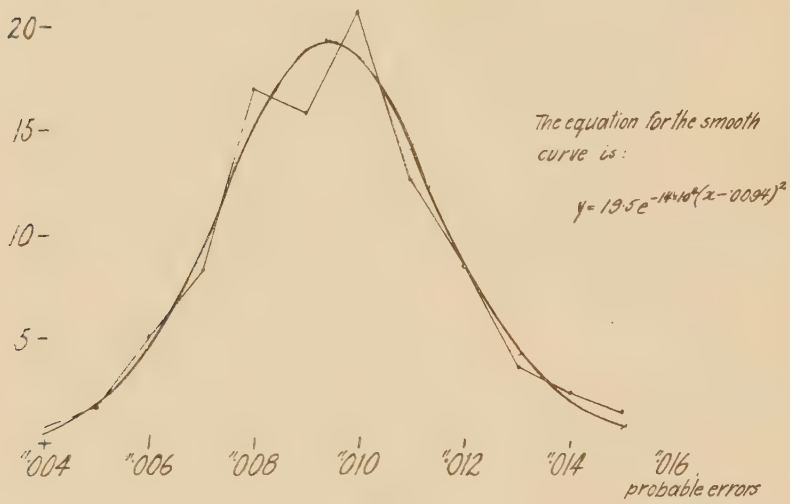


FIG. 4.

distribution, and that in every case this distribution follows the Gaussian law. To show this the parallaxes were divided in the following manner:

100	stars	with	negative	parallaxes.	
101	,,	,,	parallaxes	from	″·000 to ″·009.
104	,,	,,	,,	,,	″·010 ,, ″·019.
85	,,	,,	,,	,,	″·020 ,, ″·029.
74	,,	,,	,,	,,	″·030 ,, ″·039.

The distribution of their probable error, which is shown in figs. 3A and 3B, is about the same. Hence, combining all the parallaxes in one group we obtain the irregular curve of fig. 4; the smooth curve has for an equation $y = 19{\cdot}5e^{-14\times10^4(x-{\cdot}0094)^2}$, and represents the observed distribution of probable errors. From this the average probable error is obtained equal to $\pm$ ″·0094 $\pm$ ″·0014. Moreover, inasmuch as the distribution is perfectly symmetrical, we may proceed to compute the arithmetic mean of the probable error, together with its probable error, directly from the numerical values. This gives $\pm$ ″·0096 $\pm$ ″·0014.

The probable error of $\pm$ ″·0096 can also be obtained from the average number of plates used in the determination of each parallax without reference to the observed parallaxes. Let q represent the average

number of plates used for each parallax determination and $n = q - 3$. The value of q was found to be equal to 20. Formula *

$$\frac{\text{probable error of probable error}}{\text{probable error}} = \frac{\cdot 4769}{\sqrt{n-1}} \quad . \quad . \quad (2)$$

gives $= \pm''\cdot 0012$ for the probable error of the probable error, which agrees well with the previously computed value.

5. If the probable errors are grouped with reference to the number of plates used in their determination, the same symmetry of distribution is revealed, and the spread of this distribution agrees well with the spread computed by means of formula (2).

For example, taking all the probable errors of 84 stars having 15 or 16 plates each for their determination, we obtain for the mean probable error $\pm''\cdot 0094 \pm''\cdot 0016$, and from formula (2) for the precision of the probable error, $\pm''\cdot 0013$. The corresponding values, when 85 stars with 22 or 23 plates are used, are $\pm''\cdot 0093 \pm''\cdot 0012$, and from formula (2) $\pm''\cdot 0010$.

The writer is under obligation to Professor Eddington, who suggested this investigation and has given considerable advice during its progress.

Cambridge :
1928 *March.*

On the Application of the Brunsviga-Dupla Calculating Machine to Double Summation with Finite Differences. By L. J. Comrie, M.A., Ph.D. (Plates 4 and 5.)

Summary.—The problem to which the machine is applied is that of developing a function by continuous summation from its known finite second differences or from its second differential. The production of tables by interpolation is reduced to the same numerical procedure. Earlier attempts to produce machines for this purpose are briefly reviewed. The Nova-Brunsviga is described and illustrated, leading up to a description of the Brunsviga-Dupla. The application of the latter to the special problem mentioned in the title is treated in detail. The use of the machine for computing directly the second differences of a given function is also explained. Other applications of the machine are described to illustrate its special adaptability to scientific problems.

Statement of the Problem.—It is well known that the computation of a continuous function $y = f(x)$ at equal intervals of the independent variable x can best be tested for freedom from accidental errors by the process of differencing. This is illustrated by the following extract from an 8-decimal table of tan x for every 0·001 radian.

* Brunt, *The Combination of Observations*, p. 57.

		Differences			
x	Tan x	First	Second	Third	Fourth
0·800	1·02963856		+424		+1
		+206228		+2	
·801	1·03170084		426		−1
		206654		1	
·802	1·03376738		427		+2
		207081		3	
·803	1·03583819		430		−1
		207511		2	
·804	1·03791330		432		−2
		207943		0	
·805	1·03999273		432		+3
		208375		3	
·806	1·04207648		435		−2
		208810		1	
·807	1·04416458		436		+2
		209246		3	
·808	1·04625704		439		−2
		209685		1	
·809	1·04835389		440		+1
		210125		2	
·810	1·05045514		442		0

The differencing is usually continued until a difference is attained which oscillates in sign. The last difference is frequently formed mentally, and not written. Knowing the order of this difference (fourth in the illustration), the theoretical limit to the magnitude of the oscillations in a given order, and the degree of accuracy to be expected in the function when its manner of computation is taken into consideration, the computer is able to detect and locate accidental errors.

It will be seen that this process of computation and subsequent checking involves, first, the calculation of the function in question, and then the formation of a series of progressively diminishing differences. Existing calculating machines afford very little help in this differencing, as a computer of ordinary skill can perform the subtractions nearly as rapidly as the numbers can be set on a machine. Mathematicians have, therefore, frequently reversed the process by computing initially one of the higher order differences and then developing the function by continuous summation. A classical example of the adoption of this procedure is afforded by the *Tables du Cadastre*, computed under the direction of Prony at the end of the eighteenth century. The process offers three advantages: first, that the function initially calculated is small and slowly changing; secondly, that a calculating machine can be economically used to perform the necessary summations, and thirdly, that the control over the work is much more effective, provided occasional comparison values of the function be known or computed independently.

The use of a single machine to perform the summations necessitates the writing of the successive higher orders of differences as they are produced, while the use of a battery of machines, when such can be afforded, involves the necessity of transferring the difference developed on any machine to the setting levers of another machine. These operations take time, and introduce the risk of error, which is fatal

to the success of the method, for an error introduced at any stage causes all subsequent values to be erroneous. On this account attempts have been made to design machines capable of developing a function from a higher order difference, and these attempts will be reviewed in a later paragraph.

When computing values of a function for tabulation it is customary to compute at first, say, every fifth, tenth, or twentieth value, and to obtain the intervening values by interpolation. Instead of employing the Lagrangian formulæ, or performing each of the interpolations in full by one or other of the well-known central difference formulæ, the writer has developed a simple method (to be described fully in a later paper) whereby one of the difference columns, usually the second, of the interpolates is derived. The chief feature of the method is that these differences are in units of the last decimal of the function tabulated, *i.e.* they do not have extra guarding figures, and that their use gives exactly the same results as ordinary interpolation by the difference formula chosen, with the very satisfying check of exact reproduction of the original pivotal values. Thus in the example already quoted, if the values 1·02963856 and 206228 be known, together with the second differences 426, 427, 430 . . . it is obviously possible to produce the remaining values of $\tan x$. The chief feature of the Brunsviga-Dupla may be anticipated at this stage. If the small numbers 426, 427, 430 . . . are supplied to the machine in succession, it will produce successive values of the first difference, of $\tan x$, and even of the argument x, only one turn of the handle being necessary for each new value of $\tan x$ generated.

A similar summation process is involved in the evaluation of y when numerical values of $\frac{d^2y}{dx^2}$ are known for equal increments of x. Two initial values, either of y, or one of y and one of $\frac{dy}{dx}$, corresponding to the constants of integration, must be known. The column produced by the double summation is not y (unless the first differences of $\frac{d^2y}{dx^2}$ are constant), but is easily converted into the desired integral by means of the formulæ given in text-books. Alternatively $\frac{d^2y}{dx^2}$ may be converted into a finite second difference before summation, as was done by Cowell and Crommelin in their essay on Halley's Comet.* However, the conditions appertaining in Cowell's method, in which second differentials in three rectangular co-ordinates are required, and each can be computed only after an extrapolation for x, y, and z, would demand the simultaneous use of three Brunsviga-Duplas, one for each co-ordinate. Nevertheless, if much of this work were to be done, the saving of time and effort achieved by the machines would be sufficient to warrant their installation.

* "Investigation of the Motion of Halley's Comet," *Publikation der Astronomischen Gesellschaft XXII*, or *Appendix to Greenwich Observations for* 1909.

In physical and engineering science similar problems arise, for instance in the determination of distance travelled when the acceleration is known, or in determining bending moment from the known loading of a beam. Frequently a series of consecutive values of y is required from the equation $y = a + bx + cx^2$, in which the constant second difference is $2c$; only one turn of the handle is required for each new value.

Historical.—The first effort to produce a machine for mechanical summation was that of Charles Babbage (1791–1871), who in 1812, while still at Cambridge, conceived the idea of his Difference Engine. In 1823 the Government became interested in the project, which was strongly recommended by the Royal Society, and sums of money totalling £17,000 were voted for the construction of the machine. In 1833 the work was suspended, and after much delay Babbage was informed in 1842 that the Government regretted the necessity of abandoning the machine on account of the expense of its completion. A small portion had, however, been put together in 1833, and is now on view in the South Kensington Science Museum.*

In 1834 George Scheutz of Stockholm, having read the description of Babbage's difference engine, commenced to design a machine for the same purpose. In 1853, with his son Edward and the help of a Government grant, he was able to complete a machine, which, after exhibition at the apartments of the Royal Society in Somerset House, and in Paris, was acquired by the Dudley Observatory, Albany, N.Y. A second machine was made in Bermondsey in 1858 by Brian Donkin from Scheutz's drawings, and was used in the office of the Registrar-General to produce the voluminous tables accompanying the Third English Life Table. This machine also is now in the South Kensington Science Museum.*

In 1863 another Swedish inventor, Wiberg, successfully constructed a similar machine, which was used to calculate and cast the type for a table of 7-figure logarithms. It was presented to the Academy of Sciences, and was the subject of a eulogistic report by Delaunay.† A fourth machine was designed by George B. Grant,‡ but apparently did not pass beyond the model stage.

Some twenty-five years ago, when the need for 8-figure logarithms began to be acutely felt, the Royal Prussian Academy of Sciences of Berlin and the Imperial Academy of Sciences of Vienna commissioned Professors Bauschinger and Peters to prepare the requisite tables. Herr Hamann of Berlin-Friedenau, the inventor of the Mercedes calculating machine, constructed for them a "new machine, by means of which the function-value was to be reckoned from the second differences by summation and at once written down."§ This machine was

* *Catalogue of the Collections in the Science Museum, South Kensington.* "Mathematics. I. Calculating Machines and Instruments." London, H.M. Stationery Office, 1926.

† *Comptes Rendus*, **56**, 330 (1863).

‡ *American Journal of Science and Arts*, Third Series II, 113 (August 1871).

§ J. Bauschinger and J. Peters, *Logarithmic Trigonometrical Tables with Eight Decimal Places.* Leipzig, Engelmann, 1910.

entirely successful, and the calculations performed on it are embodied in seven volumes of the most highly-valued logarithmic tables now existing. It is sad to relate that the machine has since been stolen and that the drawings are lost, so that it cannot be reproduced.

About 1912 Mr. T. C. Hudson, of the Nautical Almanac Office, turned his attention to this problem and entered into negotiations with the Burroughs Adding Machine Company. He was able to obtain a keyboard machine capable of working in either the sexagesimal or decimal systems, of performing subtraction directly (then a new feature) and not by the addition of complements, and of printing its results. But as it would perform a single series of summations only Mr. Hudson ingeniously contrived to commence with the second difference, which was printed, together with the first difference generated. The paper was then fed through the machine a second time, the first difference being overprinted on the original printing, and the function also printed. Any error in setting the first difference during the second passage of the paper would be detected instantly by the non-correspondence of the overprinted figures with the original. This machine was exhibited at the Napier Tercentenary Exhibition * in Edinburgh in 1914, and is still in use, but with an auxiliary machine, for summations from second differences with printing of the results.

A new type of commercial Burroughs machine containing an extra register has recently appeared. It is capable of developing a function from its second differences, and of printing all the figures involved and even the argument (in red if desired) in the usual manner as exhibited on p. 448. The Brunsviga-Dupla lacks printing features, but on the other hand it costs only about one-third as much as the Burroughs, while it is also an ordinary calculator for multiplications and divisions. It must be borne in mind that the designers of these machines had in view their commercial use, and it was only by investigating in detail their mechanical possibilities that the writer was able to make the happy discovery of their suitability for this particular type of work, and general adaptability to the needs of scientific computers.

Description.—The older models of the Brunsviga Calculating Machine are well known, and do not need description here. Their principal deficiencies were the absence of tens transmission in the multiplier register, the absence of a sight dial to permit easy checking of the numbers set on the levers, the lack of a device to permit readily the making of single carriage steps, and the excessive labour of zeroising or clearing the registers by wing-nuts against the friction of the check springs.

These shortcomings are all met in the Nova-Brunsviga, which first appeared in 1926. It will be seen by reference to Plate 4 that a sight dial or check register immediately above the setting levers permits any number set to be read in one straight line. The multiplier or revolution register has been transferred from its position at the left of the moving carriage to a stationary position above and behind the setting lever

* *Napier Tercentenary Celebration Handbook*, "H.M. Nautical Almanac Office Anti-Differencing Machine," p. 127.

sight dial. A moving indicator shows the particular wheel that will be affected by a turn of the handle. This register now has tens transmission throughout, a feature that is of the utmost importance as it enables short-cutting to be used in multiplication, while still giving visible proof of the multiplier. Thus if the multiplier 29 be treated as $30 - 1$ the machine shows, not a white 3 and a red 1, but 29. It has been shown elsewhere * that the use of short-cutting reduces the operating time in multiplication by one-third.

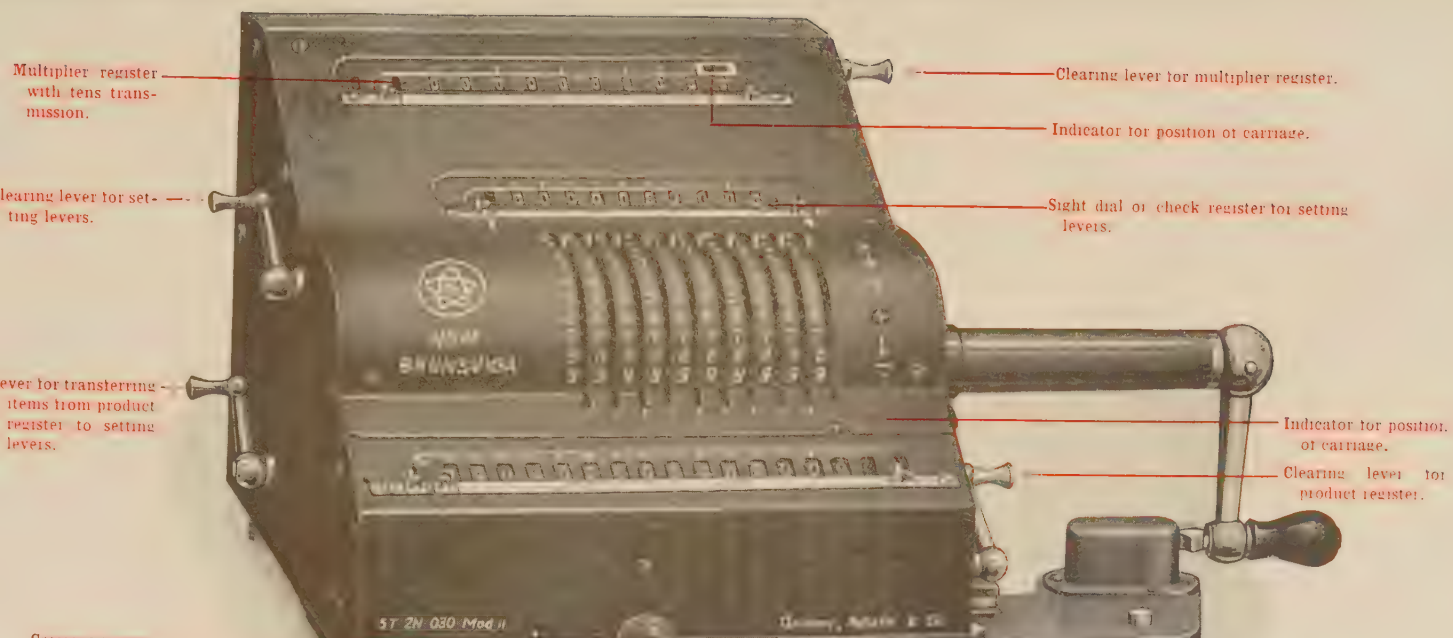
In division it is necessary that this register should record backward turns positively. In other machines having tens transmission this is accomplished by means of a lever, which is apt to cause annoyance by being left in the wrong position. In the Nova-Brunsviga the effect is secured most ingeniously by having two sets of numbers on each wheel, one in white, the other in red, each set being the complements to 9 of the other. The window through which the numbers are read is capable of a lateral movement, so that only one set—the white or the red—is visible at any moment. The intimation to the machine of the set desired—white for multiplication and red for division—is conveyed automatically by the operator by the direction of the first turn after a clearance of this register. If the first turn is forward, as is naturally the case in multiplication, the white figures are shown till the next clearance. If the first turn is backward, division is indicated and the sliding window moves to the left, revealing the red figures; at the same time, no matter in what position the carriage may be, an extra 1 is subtracted from the units wheel, thereby producing the necessary effect of converting the complements on the last wheel in use into complements to 10 instead of to 9.

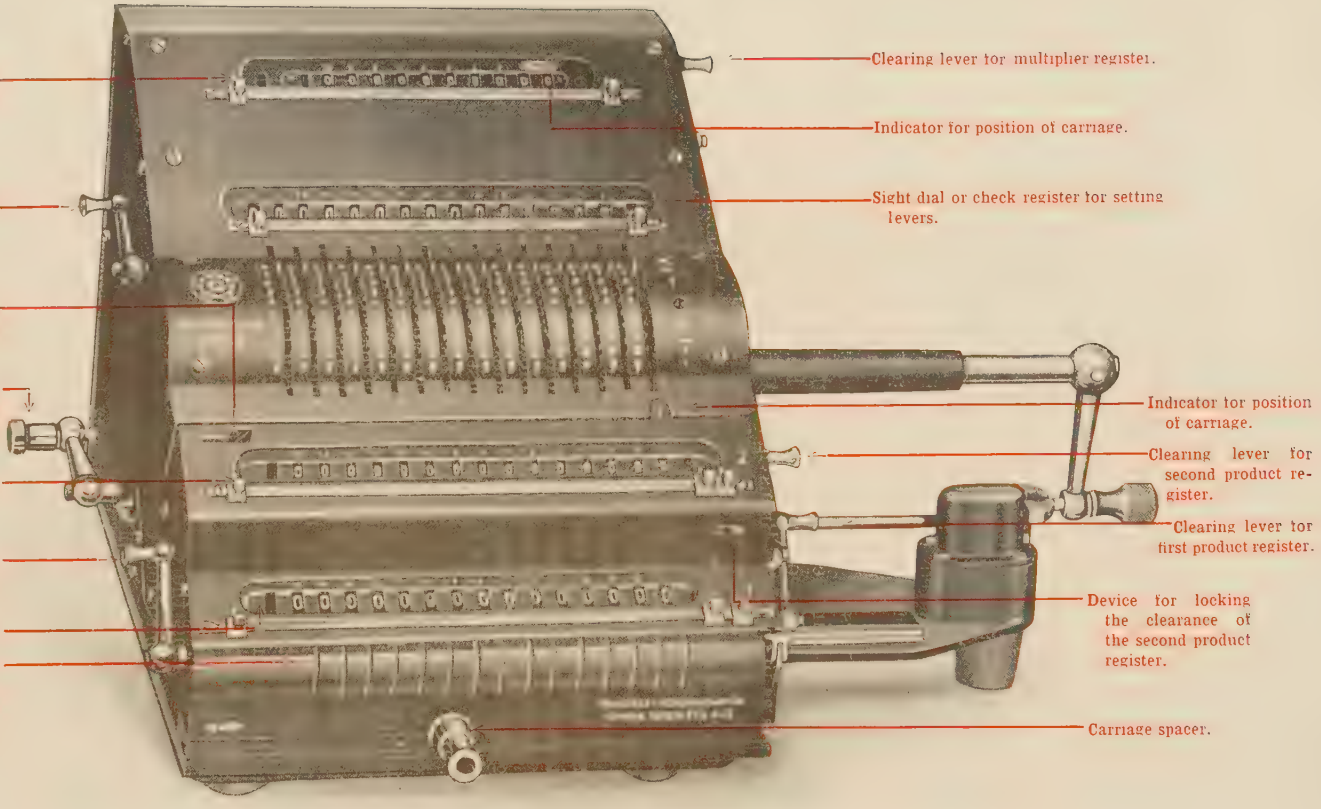
White	Red
0	9
1	8
2	7
.	.
.	.
.	.
8	1
9	0

The clearing of all registers is now effected by short levers, which are pulled through 180° and released. During this operation all tension on the setting levers or wheels that are being cleared is first removed. The carriage spacer, whilst retaining the facility of free traversing movement, permits the operator, by a simple pressure on either side, to step the carriage one place at a time, which is a great convenience.

A new and unique feature is a device for transferring figures on the product register to the setting levers, these having previously been cleared. A lever on the left of the machine is drawn forward, and displaces the carriage slightly to the left, so that the numeral wheels engage in toothed sectors on the setting levers. The carriage is held in this position while the product register is cleared in the usual way. As the numeral wheels are rotated to their zero positions they turn the setting levers through the same amount, thus effecting the desired transfer. This facility is, of course, of special use in continuous multiplication, *i.e.* $a \times b \times c \times d \times \ldots$, although other applications will be found. For instance it is easy to determine a value of y from $y =$

* L. J. Comrie, "Computing by Calculating Machines," in the *Accountants' Journal*, **45**, 42 (1927 May).


Multiplier register with tens transmission.
Clearing lever for multiplier register.
Indicator for position of carriage.
Clearing lever for setting levers.
Sight dial or check register for setting levers.
Lever for transferring items from product register to setting levers.
Indicator for position of carriage.
Clearing lever for product register.
Carriage spacer.
NOVA BRUNSVIGA
ST 2N 030 Mod II


Multiplier register with tens transmission.
Clearing lever for multiplier register.
Indicator for position of carriage.
Clearing lever for setting levers.
Sight dial or check register for setting levers.
Switch to expose the red figures in second product register.
Lever for transferring items from product registers to setting levers.
Indicator for position of carriage.
Clearing lever for second product register.
Second product register, R_2.
Clearing lever for first product register.
Lever for disconnecting the second product register.
Device for locking the clearance of the second product register.
First product register, R_1.
Thumb wheels for direct setting of figures into first product register.
Carriage spacer.

The Brunsviga-Dupla

$a + bx + cx^2$ without writing the results of any intermediate stage of the calculation.

This machine has been described at some length, partly because it is believed to be one of the best all-round calculating machines for ordinary use, and partly to prepare the reader for the real object of this paper—a description of some of the capabilities of the Brunsviga-Dupla, which appeared in the autumn of 1927, and has greater possibilities for scientific work than any other single existing machine.

It will be seen from Plate 5 that the number of setting levers on the Brunsviga-Dupla is 15. The principal feature distinguishing this machine from the Nova-Brunsviga is the addition of an extra product register, to be designated R_2, immediately below the setting levers, and which may be connected or disconnected at will by a lever on the left of the machine. When the handle is turned forward any number on the setting levers is necessarily added to the principal product register R_1, situated in the front of the machine, and is also added to R_2 if it is connected. The contents of either R_1 or R_2 may be transferred to the levers as in the Nova-Brunsviga. A number may be introduced directly into R_1 by means of thumb wheels in the front of the carriage. This feature restores one of the advantages of the Arithmometer type of machine, which, up to the present, has been absent on barrel-type machines. Its principal application is in the division of several numbers by a constant divisor, which can be left undisturbed on the setting levers. If the divisor remained constant for many divisions it would, of course, become worth while to form its reciprocal and resort to multiplication, but this is not the case for only three or four divisions.

The register R_2 is provided with two-colour figures as in the multiplier register. A switch, which can be operated only when the carriage is in its left-hand position (as in the illustration) moves the sliding window and exposes the red figures. The first turn after moving the switch can only be a backward one, as this introduces the necessary compensation in the units wheel. When R_2 shows red figures a forward turn of the handle will add the number on the setting levers to R_1 and subtract it from R_2, and vice versa. In obtaining x from $x = a \pm b \pm c \pm \ldots$ both registers could be employed, with R_2 showing red; if the answer is positive it can be read directly from R_1 and will be shown as a complement in R_2; if negative it can be read directly from R_2 and will appear in R_1 as a complement. As the normal function of R_2 is that of a storing or accumulating register, a device is provided for locking the clearance to prevent unintentional clearing during a lengthy continuous calculation.

Applications.—As a simple illustration of the use of two registers consider the problem frequently arising in least squares, correlation, and curve-fitting work of determining the products x_1y_1, x_2y_2 . . . of two columns of numbers x_1, x_2, . . . and y_1, y_2, . . . as well as x_1^2, x_2^2, . . . y_1^2, y_2^2, . . . and the sums $[x]$, $[y]$, $[xy]$, $[x^2]$, and $[y^2]$. If, as is often the case, the numbers involved are positive and less than 1000 the large capacity of this machine will permit three operations to be performed simultaneously. Suppose columns 1 to 6 are devoted

to y (which is set at the right of its allotted field), columns 7 to 12 to x, and that the number 1 is set on setting lever 13. If the handle is turned for multiplication by x the registers (R_2 can be locked) will show

R_2	x_1	x_1^2	x_1y_1
R_1	x_1	x_1^2	x_1y_1

After noting x_1^2 and x_1y_1 the register R_1 is cleared, the setting is changed to 1, x_2, y_2 and multiplication by x_2 is performed, showing on the registers

R_2	$x_1 + x_2$	$x_1^2 + x_2^2$	$x_1y_1 + x_2y_2$
R_1	x_2	x_2^2	x_2y_2

When all the numbers are exhausted R_2 will show $[x]$, $[x^2]$, and $[xy]$. A repetition with the same settings but multiplication by y will produce $[y]$, $[xy]$, and $[y^2]$, so that a partial check is afforded by the agreement of the two values of $[xy]$. If $[xy]$ is wanted in a case where both positive and negative signs occur, the accumulation can be done on R_1, and the individual products read from R_2, which would be switched to red for negative products.* Alternatively, if R_2 is used for the accumulation it must be disconnected while a negative xy is formed in R_1 by forward turning; this product is then transferred to the levers, R_2 is connected, and the handle turned backward. Modifications to suit the particular requirements of specific problems will suggest themselves to the experienced computer.

The ability of the machine to perform double summation from second differences involves the two-register feature, including the two-colour wheels in R_2 and the possibility of connecting or disconnecting at will, the transfer feature, and the thumb wheels for direct setting in R_1. Other calculating machines having two registers cannot be used for this purpose as they lack the mechanical transfer, which is an exclusive Brunsviga feature. The multiplier register is devoted to the argument, R_2 to the function being developed, the setting levers to the first difference Δ', while the second difference Δ'' is entered into R_1 by the thumb wheels.

Case I

Arg.	Function	Δ'	Δ''
1	**12345**		
		+287	
2	12632		**+32**
		+319	
3	12951		**+34**
		+353	
4	13304		**+37**
		+390	
5	13694		**+39**
		+429	
6	14123		**+40**
		+469	
7	14592		

The numbers initially known are those in bold type. 12345 is set on the levers and the handle turned. R_1 is cleared, leaving the argument 1 in the multiplier register and 12345 in R_2, which may be locked. 287 is set on the levers and 32 in R_1. A forward turn of the handle will increase the argument to 2, and will add 287 to 12345 in R_2 to form the new value 12632 of the function and will also add 287 to 32 in R_1 to form the next value 319 of Δ'. To continue the process the levers are cleared and the contents—319,

* When $[xy]$ contains positive and negative products it is not possible to form $[x^2]$ and $[y^2]$ at the same time as $[xy]$.

or in general Δ'—of R_1 are transferred by the transfer device to the levers. This has cleared R_1, into which the new value 34 of Δ'' is turned by the thumb wheels. The work now progresses continuously in this manner. The first difference is never handled, nor need it be recorded. The cycle of operations is—set Δ'', turn handle forward, read and record the result in R_2, clear levers and transfer R_1. To one who has had to do much of this work by hand, or on two separate machines, or in two separate operations on one machine, the simplicity of the process is really fascinating.

The question naturally arises as to whether the machine can handle cases in which either Δ' or Δ'' or both are negative. If the function is negative we may consider all the signs reversed. Four distinct cases can arise, and it is worth while to put on record here the treatment required by each, because of the interest of the principles involved. The conditions which it is desirable to fulfil are—

(1) The actual value of Δ'', and never its complement, is to be set in R_1.
(2) The handle is to be turned forward always. This avoids confusion; the argument can be made to increase or decrease at will by using white or red figures respectively in the multiplier register.
(3) The mode of operating (*i.e.* the cycle mentioned above) should be the same for all cases.
(4) If a change from one case to another occurs during the work it should be possible to make the necessary changes without a complete resetting of the machine.

Case II

Arg.	Function	Δ'	Δ''
1	**12345**		
		+287	
2	12632		**−32**
		+255	
3	12887		**−34**
		+221	
4	13108		**−37**
		+184	
5	13292		**−39**
		+145	
6	13437		**−40**
		+105	
7	13542		

Case III

Arg.	Function	Δ'	Δ''
1	**12345**		
		−287	
2	12058		**+32**
		−255	
3	11803		**+34**
		−221	
4	11582		**+37**
		−184	
5	11398		**+39**
		−145	
6	11253		**+40**
		−105	
7	11148		

Case IV

Arg.	Function	Δ'	Δ''
1	**12345**		
		−287	
2	12058		**−32**
		−319	
3	11739		**−34**
		−353	
4	11386		**−37**
		−390	
5	10996		**−39**
		−429	
6	10567		**−40**
		−469	
7	10098		

The first principle invoked is that of subtracting by adding complements. Thus in Case III if we add 99 . . . 99713 to 12345 we shall obviously produce the same effect as by subtracting 287, while the addition of this number in R_1 to 32 will produce 99 . . . 99745, which when added to a number is equivalent to the subtraction of 255. Again, if we use the red figures in R_2 and turn the handle forward, the result

is a subtraction of the number on the levers from R_2 and its addition to R_1; this is required in Case IV. Finally it will be seen that Case II requires the use of red figures in R_2 and of complements on the setting levers. Summarising:—

Case	Function	Δ'	Δ''	Initial Setting of	
				R_2	Levers
I	+	+	+	White	Direct
II	+	+	−	Red	Complementary
III	+	−	+	White	Complementary
IV	+	−	−	Red	Direct

In practice reference to this table is unnecessary, because the two following simple and easily remembered rules suffice:—

(1) If Δ'' is positive (normal) R_2 is white (normal): if Δ'' is negative R_2 is red.
(2) If the product of Δ' and Δ'' is positive (normal) the lever setting is direct (normal): if this product is negative the lever setting is complementary.

A complementary lever setting is easily made by setting the direct value on the levers, turning the handle backwards, and then transferring R_1 to the levers, although the more advanced operator would simply increase the setting of the first argument by 2, set the actual value of the initial Δ' on the levers, and make the first turn only backward instead of forward. To avoid the heavier turning that would result when most of the setting levers bore 9s the work may be done further to the left of the machine. Thus Cases II and III could be done with the 5 left-hand levers, the remaining 10 being inactive. It might be desirable to know at each stage the value of Δ' when it is being handled as a complement; if so, after transferring R_1 to the levers the handle should be turned backward, when the desired value will appear in R_1. A forward turn of the handle clears R_1 and the work proceeds as before.

If in the course of working a change of sign occurs in the function or in one of the differences, there will be a transition from one of the above-mentioned cases to another. The following general rules should be borne in mind:—

(*a*) If the change in case is due to a change in the sign of the function, R_2 only has to be changed, and the necessity for the change will be indicated by the appearance of a string of 9s in R_2.
(*b*) If the change in case is due to a change in the sign of Δ' (which as a matter of fact can only occur in cases where Δ' is set as a complement) no change of procedure is necessary. The complement becomes a direct number and the work proceeds.
(*c*) If the change in case is due to a change in the sign of Δ'' both R_2 and the levers must be changed.

The following two cases fall under Rule (*a*):—

(a_1) *Case III to Case IIA*, i.e. +−+ *to* −−+

Arg.	Function	Δ′	Δ″
1	+100		
		−46	
2	+ 54		+ 7
		−39	
3	+ 15		+11
		−28	
4	− 13		+12
		−16	
5	− 29		+11
		− 5	
6	− 34		

When complements first appear in R_2 switch over to red figures, give a reverse and forward turn in succession and proceed.

(a_2) *Case IV to Case IA*, i.e. +−− *to* −−−

Arg.	Function	Δ′	Δ″
1	+100		
		−38	
2	+ 62		− 7
		−45	
3	+ 17		− 9
		−54	
4	− 37		−12
		−66	
5	−103		−15
		−81	
6	−184		

When complements appear in R_2 transfer R_2, increase lever setting by 1 and turn forward. Disconnect R_2 and reverse. Connect R_2 and proceed.

The following two cases depend on Rule (*b*):—

(b_1) *Case II to Case IV*

Arg.	Function	Δ′	Δ″
1	1000		
		+20	
2	1020		− 7
		+13	
3	1033		−10
		+ 3	
4	1036		−14
		−11	
5	1025		−17
		−28	
6	997		

(b_2) *Case II to Case I*

Arg.	Function	Δ′	Δ″
1	1000		
		−20	
2	980		+ 7
		−13	
3	967		+10
		− 3	
4	964		+14
		+11	
5	975		+17
		+28	
6	1003		

In both cases the operation is continuous.

The following four cases depend on Rule (*c*):—

(c_1) *Case I to Case II*

Arg.	Function	Δ′	Δ″
1	100		
		+20	
2	120		+7
		+27	
3	147		+2
		+29	
4	176		−3
		+26	
5	202		−8
		+18	
6	220		

When Δ″ becomes negative leave R_1 and transfer * R_2. Switch R_2 to red and reverse. Disconnect R_2 and turn forward. Transfer * R_1 and reverse. Clear levers, turn forward, connect R_2 and proceed.

* It is understood that the setting levers must be cleared before a transfer can be effected.

(c_2) *Case II to Case I*

Arg.	Function	Δ′	Δ″
1	**1000**		
		+100	
2	1100		**−7**
		+ 93	
3	1193		**−2**
		+ 91	
4	1284		**+3**
		+ 94	
5	1378		**+8**
		+102	
6	1480		

When Δ″ becomes positive leave R_1 and transfer * R_2. Increase the lever setting by 1 and reverse. Disconnect R_2 and turn forward. Transfer * R_1 and reverse. Clear levers, turn forward, connect R_2 and proceed.

(c_3) *Case III to Case IV*

Arg.	Function	Δ′	Δ″
1	**1000**		
		−46	
2	954		**+ 9**
		−37	
3	917		**+ 2**
		−35	
4	882		**− 5**
		−40	
5	842		**−12**
		−52	
6	790		

When Δ″ becomes negative leave R_1 and transfer * R_2. Switch R_2 to red and reverse. Disconnect R_2 and turn forward. Transfer * R_1 and reverse. Clear levers, turn forward, connect R_2 and proceed.

(c_4) *Case IV to Case III*

Arg.	Function	Δ′	Δ″
1	**1000**		
		−84	
2	916		**− 9**
		−93	
3	823		**− 2**
		−95	
4	728		**+ 5**
		−90	
5	638		**+12**
		−78	
6	560		

When Δ″ becomes positive leave R_1 and transfer * R_2. Increase lever setting by 1 and reverse. Disconnect R_2 and turn forward. Transfer * R_1 and reverse. Clear levers, turn forward, connect R_2 and proceed.

Although this may sound complicated to a person who has not seen and handled the machine it is not really so. As a test the four cases and the eight possible transferences of cases were shown to a friend accustomed to machines; after working once through each of the examples here given he was able to discard notes completely and operate entirely from the principles involved.

One other important use of the machine that should be mentioned is its ability to difference a function, producing second differences directly. If the differences are required for checking purposes only, the more quickly the higher order differences can be obtained the better. Also the modern tendency in statistical tables is to use a rather wide interval and to publish even-order differences with the tables, subsequent interpolation being performed by Everett's formula.

* It is understood that the setting levers must be cleared before a transfer can be effected.

The use of an ordinary machine to obtain consecutive orders of differences (which must *all* be written down) is so laborious that it saves little, if any, time and effort. It is well known that with consecutive values $a, b, c, d, e, f, g, \ldots$ of a function the second differences are $a - 2b + c, b - 2c + d, \ldots$ and this formula is frequently used with ordinary machines. It requires that each of the numbers $c, d, e, f, \ldots$ should be set twice. On the Brunsviga-Dupla each number is set once only. After the first few initial operations the work goes in cycles of six simple operations for each new second difference, the results being read from R_1 and R_2 alternately.

No. in Cycle	Operation	Levers	R_1 contains	R_2 contains
..	Set	a	0	0
..	Turn	a	a	a
..	Set	b	a	a
..	Turn back 3 times	b	$a-3b$	$a-3b$
..	Clear R_2	b	$a-3b$	0
..	Turn	b	$a-2b$	b
1	Set	c	$a-2b$	b
2	Turn	c	$a-2b+c$	$b+c$
3	Read and record result	c	Δ''	$b+c$
4	Turn back 4 times	c	$a-2b-3c$	$b-3c$
5	Clear register just read	c	0	$b-3c$
6	Turn	c	c	$b-2c$
1	Set	d	c	$b-2c$

If the second difference is negative the instruction "turn" must read "reverse or turn backward" and "turn back four times" must read "turn forward four times." The second differences thus found may, of course, be subjected to the same process to obtain fourth differences, and so on.

Conclusion.—This paper does not by any means exhaust the scientific possibilities of this remarkable machine, which is now within the reach of all scientific institutions. As it becomes more widely known and falls into the hands of workers in different fields it is certain to yield new possibilities. Someone will want x^2 and $1 - x^2$; the machine will furnish both values in one operation. Examples could be multiplied indefinitely, but perhaps those already given will inspire some at least of our computers to further study of the machine.

The Fourier Expansions of $(1 + a^2 - 2ak\cos\psi)^{-s}$.

By Ernest W. Brown, Sc.D., F.R.S.

1. The expansions of this function for $s = \frac{1}{2}, \frac{3}{2}, \ldots$, with $k = 1$, constitute the first step in the development of the disturbing function for the planetary theory. The following rapid method of computing the coefficients for any value of a, without the use of tables of elliptic

functions, appears to be new. The formulæ for $k \neq 1$ also appear to be new.

What we need are the coefficients of $\cos i\psi$ in the expansion, for $i = 0, 1, \ldots$, with $s = \frac{1}{2}, \frac{3}{2}, \ldots$. It is well known that if we have calculated the coefficients for any two consecutive values of i for a given s, the coefficients for any other values of i and s can be obtained by recurrence formulæ. The usual plan is to find those for $i = 0, 1$, with $s = \frac{1}{2}$ either from a series or by means of a table of elliptic functions. I have pointed out * that the work is easier and that there is less loss of accuracy in using the recurrence formulæ, if we start with some high value of i and work backwards to $i = 0$. As a convenient place, I start with $i = 10, 11$, from which we can work backwards to $i = 0$, or forward if necessary.

2. We have,† when $k = 1$,

$$(1 + \alpha^2 - 2\alpha\cos\psi)^{-s} = \tfrac{1}{2}\sum_{-\infty}^{+\infty} \frac{s(s+1)\ldots(s+i-1)}{i!} \frac{\alpha^i}{(1-\alpha^2)^s} g_s^{(i)} \cos i\psi,$$

where

$$g_s^{(i)} = 1 \quad \frac{s}{1}\cdot\frac{1-s}{i+1}p + \frac{s(1+s)}{1\cdot 2}\cdot\frac{(1-s)(2-s)}{(i+1)(i+2)}p^2 \ldots, \quad p = \frac{\alpha^2}{1-\alpha^2}.$$

This last series is divergent for $p>1$. But since the function which it represents has no singular point between $0<\alpha<1$, the method of analytic continuations can be used to obtain series which are convergent for $p>1$. The above may be regarded as an expansion about the point $p = 0$. I have deduced expansions about the points $p = 1, \frac{1}{2}$, which turned out to converge even more rapidly that those for $p = 0$, and which are sufficient for all the needs of the planetary theory.

3. Put

$$(1 + \alpha^2 - 2\alpha\cos\psi)^{-\frac{1}{2}} = \tfrac{1}{2}\sum_{-\infty}^{+\infty} \beta_{\frac{1}{2}}^{(i)} \cos i\psi \qquad . \quad (3.1)$$

then the two required coefficients are given by the following formulæ:—

$\frac{1}{2}\beta_{\frac{1}{2}}^{(10)} = 11\cdot 13\cdot 17\cdot 19\left(\frac{\alpha}{4}\right)^9\sqrt{p}$, multiplied by one of the series.

(1).	(½).	(0).
+0·97912 120	+0·98913 047	+1·00000 000
− ·01926 520$(p-1)$	− ·02082 065$(p-\frac{1}{2})$	− ·02272 727p
+ ·00141 681$(p-1)^2$	+ ·00171 01$(p-\frac{1}{2})^2$	+ ·00213 068p^2
− ·00016 594$(p-1)^3$	− ·00023 04$(p-\frac{1}{2})^3$	− ·00034 15p^3
+ ·00002 515$(p-1)^4$	+ ·00004 1$(p-\frac{1}{2})^4$	+ ·00007 47p^4
− ·00000 451$(p-1)^5$	− ·00000 9$(p-\frac{1}{2})^5$	− ·00002 0p^5
+ ·00000 091$(p-1)^6$	+ ·00000 2$(p-\frac{1}{2})^6$	+ ·00000 6p^6
− ·00000 020$(p-1)^7$		
+ ·00000 005$(p-1)^8$		
− ·00000 001$(p-1)^9$		

* *Action of the Planets on the Moon*, Camb. Univ. Press, 1908.

† Tisserand, *Méc. Cél.*, vol. i., chap. 17.

$\frac{1}{2}\beta_{\frac{1}{2}}^{(11)} = 2 \cdot 13 \cdot 17 \cdot 19 \cdot 21\left(\frac{\alpha}{4}\right)^{10}\sqrt{p}$, multiplied by one of the series.

(1).	(½).	(0).
+0·98074 527	+0·99000 356	+1·00000 000
− ·01786 480$(p-1)$	− ·01920 796$(p-\frac{1}{2})$	− ·02083 333p
+ ·00123 022$(p-1)^2$	+ ·00146 840$(p-\frac{1}{2})^2$	+ ·00180 288p^2
− ·00013 597$(p-1)^3$	− ·00018 54$(p-\frac{1}{2})^3$	− ·00026 83p^3
+ ·00001 956$(p-1)^4$	+ ·00003 11$(p-\frac{1}{2})^4$	+ ·00005 48p^4
− ·00000 335$(p-1)^5$	− ·00000 7$(p-\frac{1}{2})^5$	− ·00001 4p^5
+ ·00000 065$(p-1)^6$	+ ·00000 2$(p-\frac{1}{2})^6$	+ ·00000 4p^6
− ·00000 014$(p-1)^7$		
+ ·00000 003$(p-1)^8$		
− ·00000 001$(p-1)^9$		

The series admit of obvious tests which have been applied.

These results are sufficient to obtain the two coefficients to *eight* significant figures for any value of α up to $\alpha = 0{\cdot}816$, as appears from the following corresponding values of α, $p = \alpha^2/(1-\alpha^2)$:—

$\alpha =$	0·88	0·82	0·80	0·70	0·60	0·50	0·40	0·30
$p =$	3·00	2·00	1·78	0·96	0·56	0·33	0·19	0·10.

For values of $p>0{\cdot}75$ we use the first series ; for those between 0·75, 0·25, the second series ; and for those less than 0·25, the third series. It is rare for values of α greater than 0·8 to be needed, but even for $\alpha = 0{\cdot}88$ the first series gives the coefficients to six significant figures.

4. For $i = 9, 8 \ldots, 0$, we make use of the recurrence formula,

$$\beta_{\frac{1}{2}}^{(i-1)} = \epsilon\beta_{\frac{1}{2}}^{(i)} - \beta_{\frac{1}{2}}^{(i+1)} + \frac{1}{2i-1}\left(\epsilon\beta_{\frac{1}{2}}^{(i)} - 2\beta_{\frac{1}{2}}^{(i+1)}\right), \quad \epsilon = \frac{1}{\alpha} + \alpha \qquad (4.1)$$

which requires only one operation, the multiplication by ϵ, with a non-integral number. A loss of accuracy of less than one significant figure is produced by the complete process to $i = 0$, in any coefficient.

5. For $s>\frac{1}{2}$, it is convenient to operate with

$$\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\ldots\cdot(s-1)\alpha^{s-\frac{1}{2}}}{(1+\alpha^2-2\alpha\cos\psi)^s} \equiv \tfrac{1}{2}\sum_{-\infty}^{+\infty}\beta_s^{(i)}\cos i\psi \,. \qquad (5.1)$$

which differs from the formula of Leverrier only in the factor

$$\tfrac{1}{2}, \tfrac{3}{2} \ldots (s-1).$$

Any *two* values, say those for $i = 10, 11$, are computed from

$$\left.\begin{aligned}\tfrac{1}{2}\left(\beta_{s+1}^{(i)} + \beta_{s+1}^{(i+1)}\right) &= \frac{(i+s)\beta_s^{(i)} - (i-s+1)\beta_s^{(i+1)}}{2(1-\alpha)^2}\cdot\alpha \\ \tfrac{1}{2}\left(\beta_{s+1}^{(i)} - \beta_{s+1}^{(i+1)}\right) &= \frac{(i+s)\beta_s^{(i)} + (i-s+1)\beta_s^{(i+1)}}{2(1+\alpha)^2}\cdot\alpha\end{aligned}\right\} \qquad (5.2)$$

These are used in succession for $s = \frac{1}{2}, \frac{3}{2} \ldots$. A loss of accuracy of not more than one significant figure occurs after three operations.

For the remaining values of i with $s = \frac{1}{2}, \frac{3}{2} \ldots$, we have the formula,

$$\beta_{s+1}^{(i-1)} = \beta_{s+1}^{(i+1)} + i\beta_s^{(i)} \quad . \quad . \quad . \quad (5.3)$$

The whole of the method given here is, in fact, based on the manner in which this test formula lends itself to easy computation. Either of the formulæ (5.2) will serve as a check if we put $i = 0$ therein. A gain of accuracy is produced by the use of (5.3).

6. For the derivatives with respect to α, we put

$$\beta_{s,p}^{(i)} = \frac{\alpha^p}{p!}\left(\frac{d}{d\alpha}\right)^p \beta_s^{(i)}.$$

The first derivatives are computed from *

$$\beta_{s,1}^{(i)} = \left(\frac{1}{\alpha} - \alpha\right)\beta_{s+1}^{(i)} - \tfrac{1}{2}\beta_s^{(i)} \quad . \quad . \quad . \quad (6.1)$$

for all values of i with $s = \frac{1}{2}$, and for *two* values of i with $s = \frac{3}{2}, \frac{5}{2} \ldots$.

The same set of values of i, s are used in the computation of the second and higher derivatives from the formula.

$$p(p+1)\beta_{s,p+1}^{(i)} = 4s\alpha\left(\beta_{s+1,p-1}^{(i)} + \beta_{s+1,p-2}^{(i)}\right) - p(2p-2s)\beta_{s,p}^{(i)} + \{i^2 - (p-s-\tfrac{1}{2})^2\}\beta_{s,p-1}^{(i)} \quad (6.2)$$

This serves for $p=1$ if we put $\beta_{s+1,-1}^{(i)} = 0$.

The derivatives for the remaining values of i, s are computed from

$$\beta_{s+1,p}^{(i-1)} = \beta_{s+1,p}^{(i+1)} + i\beta_{s,p}^{(i)} \quad . \quad . \quad . \quad (6.3)$$

If it be not desired to compute complete tables of the derivatives, the formulæ (6.1), (6.2) will give them for any i as soon as the tables of § 5 have been computed. When tables are formed they can be used as checks if we put $i = 0$ in them.

7. It is not difficult to so alter these results as to make them applicable to the expansion of $(1 - 2\alpha k\cos\psi + \alpha^2)^{-s}$. For this purpose we put

$$1 + \alpha^2 - 2\alpha k\cos\psi \equiv C(1 + \alpha'^2 - 2\alpha'\cos\psi),$$

so that

$$\alpha' = -\tfrac{1}{2}\epsilon_1 + \tfrac{1}{2}\sqrt{\epsilon_1^2 - 4}, \qquad C = \frac{k\alpha}{\alpha'}, \qquad \epsilon_1 = \frac{1}{k}\left(\frac{1}{\alpha} + \alpha\right).$$

We compute

$$\beta_{\frac{1}{2}}^{(10)}, \qquad \beta_{\frac{1}{2}}^{(11)}$$

* On p. 32 of the *Action of the Planets on the Moon*, the first formula of (30) is erroneous: the factor $1-\alpha^2$ should be changed to $1/\alpha-\alpha$. The factor s there present disappears with the notation of this paper.

with

$$C^{-\frac{1}{2}}(1 + a'^2 - 2a'\cos\psi)^{-\frac{1}{2}} = \tfrac{1}{2}\Sigma\beta^{\frac{1}{2}}_{(i)}\cos i\psi.$$

The remainder of the $\beta^{(i)}_{\frac{1}{2}}$ are computed by means of (4.1), with ϵ_1 replacing ϵ therein, so that the change to a' is only needed in finding the coefficients for $i = 10, 11$.

For values of s greater than $\frac{1}{2}$, we write

$$\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\ldots\cdot(s-1)(ak)^{s-\frac{1}{2}}}{(1 + a^2 - 2ak\cos\psi)^s} \equiv \tfrac{1}{2}\sum_{-\infty}^{+\infty}\beta^i_s\cos i\psi,$$

instead of (5.1). The formulæ (5.2) are changed by putting in them

$$\frac{ak}{1 + a^2 \pm 2ka} \quad \text{for} \quad \frac{a}{(1 \pm a)^2}$$

respectively.

The recurrence formula (5.3) is unchanged.

These changes do not produce any increase in the labour of computation.

8. For the derivatives with respect to a, we make the following changes :—

In (6.1) replace $\left(\frac{1}{a} - a\right)$ by $\frac{1}{k}\left(\frac{1}{a} - a\right)$.

In (6.2) replace a by $\frac{a}{k}$ and add $\left(4 - \frac{4}{k^2}\right)\beta^{(i)}_{s+2,\,p-1}$ to the right member.

(6.3) is unchanged.

The only portion needing additional computation is the term added to (6.2), and this term has the small factor $4 - 4k^{-2}$.

9. This extension of the ordinary form can be used to abbreviate the calculation of the perturbations, especially when the inclination i is large, by putting $k = \cos^2\frac{1}{2}i$. In the usual method, slow numerical convergence occurs chiefly because each coefficient is expanded in powers of $\sin^2\frac{1}{2}i$, and this expansion is avoided here. There is, of course, a set of terms expanded in powers of this parameter, but they contain the longitude of the node, and are usually less important than those considered here. It is true that the derivative with respect to i is somewhat more complicated, but the perturbations which depend on this derivative are usually small and are not required to possess high accuracy. The gain is in the long-period terms in the mean longitude, and even for them we rarely need to use anything but the mean value of the inclination which can thus be inserted at the outset.

Part of the gain in accuracy which the method gives is seen in the coefficient $4 - 4k^{-2}$ of the term additional to (6.2). When $i = 25°$ this coefficient is 0·25, when $i = 30°$ it is 0·57.

10. The proofs of the various formulæ used here are to be found, with obvious modifications, either in Tisserand (*l.c.*) or in Sect. III. of

my *Action of the Planets on the Moon*. It is interesting, however, to note that the formula of § 2 which Tisserand deduces from the hypergeometric series can be obtained directly by writing $2\cos\psi = x + x^{-1}$, and

$$(1-2\alpha\cos\psi+\alpha^2)^{-s} \equiv \frac{(1-\alpha x)^{-s}}{(1-\alpha^2)^s}\left(1-\frac{\alpha}{1-\alpha^2}\cdot\frac{1-\alpha x}{x}\right)^{-s}$$

$$= \frac{1}{(1-\alpha^2)^s}\sum_j \binom{-s}{j}\left(\frac{-\alpha}{1-\alpha^2}\right)^j \frac{(1-\alpha x)^{j-s}}{x^j}$$

where $\binom{p}{q}$ denotes the coefficient of y^q in the binomial expansion of $(1+y)^p$. The coefficient of x^i in this expansion is

$$\frac{\alpha^i}{(1-\alpha^2)^s}\sum_j \binom{-s}{j}\binom{j-s}{i+j}\left(\frac{\alpha}{1-\alpha^2}\right)^j$$

which is the same as that in § 2.

The formulæ of § 6 are modifications, due to the presence of (i), of those on p. 32 of the volume referred to, and are proved in the same manner. It may be noted that the first formulæ of (30) and (36) are the only ones needed in the planetary theory.

11. The computation of the series in § 3 was made by finding $g_{\frac{1}{2}}^{(10)}$, $g_{\frac{1}{2}}^{(11)}$ as series in powers of p to p^{35} with an estimate, easily made, for the remainder. These gave nearly twelve places of decimals and ten places in their first derivatives with respect to p when $p = 1$ and more when $p = 0{\cdot}5$.

If

$$g_{\frac{1}{2}}^{(i)} \equiv 1 + \Sigma a_n p^n \equiv b_0 + \Sigma b_n (p-1)^n \equiv c_0 + \Sigma c_n (p-0{\cdot}5)^n,$$

these computations gave b_0, b_1, c_0, c_1. Recurrence formulæ to obtain the remaining coefficients were found by means of the hypergeometric series,

$$\mathrm{F(A, B, C, } x) = 1 - \frac{\mathrm{A}\,.\,\mathrm{B}}{1\,.\,\mathrm{C}}x + \frac{\mathrm{A(A+1)}}{1\,.\,2}\cdot\frac{\mathrm{B(B+1)}}{\mathrm{C(C+1)}}x^2 \ldots .$$

which satisfies the differential equation,

$$\mathrm{ABF} + \{\mathrm{C} + x(\mathrm{A}+\mathrm{B}+1)\}\frac{\partial \mathrm{F}}{\partial x} + (x+x^2)\frac{\partial^2 \mathrm{F}}{\partial x^2} = 0.$$

The series for $g_{\mathrm{s}}^{(i)}$ is of this form if $\mathrm{A} = s$, $\mathrm{B} = 1-s$, $\mathrm{C} = i+1$, $x = p$. When $s = \frac{1}{2}$, the differential equation becomes

$$\tfrac{1}{4}\mathrm{F} + (i+1+2p)\frac{\partial \mathrm{F}}{\partial p} + (p+p^2)\frac{\partial^2 \mathrm{F}}{\partial p^2} = 0.$$

If we put $p \equiv 1 + q \equiv 0{\cdot}5 + q'$, successively, and note that $\partial/\partial p = \partial/\partial q = \partial/\partial q'$, we can substitute the two latter series for $g_{\frac{1}{2}}^{(i)}$, and,

by equating to zero the coefficients of powers of q or q', obtain the recurrence formulæ,

$$(2n+2)b_{n+1} + (3n+i)b_n + (n-1)\left\{1 + \frac{1}{4n(n-1)}\right\}b_{n-1} = 0,$$

$$(3n+3)c_{n+1} + (8n+4i)c_n + (4n-4)\left\{1 + \frac{1}{4n(n-1)}\right\}c_{n-1} = 0.$$

A loss of accuracy results which reduces the number of places of decimals in the last coefficient required to about eight.

12. It is a simple matter to deduce series for $p - \lambda$ where $\lambda = 0{\cdot}1$, $0{\cdot}2$, ... to $2{\cdot}0$. If this be done, not more than four terms in any series will be required, since in any application, that series can be used for which $p - \lambda \leqslant 0{\cdot}05$.

If the problem does not require values of i as far as $i = 11$, we compute only the $\beta_{\frac{1}{2}}^{(i)}$ as far as this value. All the remaining work can be carried out with the highest value of i needed.

Yale University :
1928 *January* 19.

The Shortt Clocks at the Royal Observatory, Greenwich. By J. Jackson, M.A., D.Sc., and W. Bowyer.

(*Communicated by the Astronomer Royal.*)

1924 *November*–1926 *June*.—The Clock Shortt 3 was delivered at the Royal Observatory on 1924 November 21. The free pendulum was set up in a cellar in the basement of Flamsteed House where the temperature was not subject to rapid changes. The slave clock associated with it was installed in the clock room in the old main building. During the ensuing month trials were made to find the variation of the rate and arc with pressure, and the behaviour of the clock was so good that at the end of December the rate was reduced to be nearly zero, and the clock was introduced as standard sidereal clock on 1925 January 1.

The clock rate, however, steadily increased, as shown in the following table, the rates being referred to true sidereal time, which differs from mean sidereal time by, at most, $\pm 8{\cdot}003$ daily.

The pressure in the clock case was 31·0 mm. at the beginning of 1925 January. It gradually increased through leakage to 34·5 mm. on April 1, when it was reduced to 25·4 mm. in an attempt to reduce the rate. The pressure again increased to 27·4 mm. by the end of June, and in a further effort to reduce the rate the pressure was reduced to 10·2 mm. Thereafter the pressure was not deliberately altered, but by 1926 June it had increased to 20·0 mm.

From the table it will be seen that the daily losing rate increased by about $0^s{\cdot}04$ per month up till 1925 September, when the rate of

increase fell to about one-third of this amount. The reduction of the pressure by 9 mm. from 34 mm. to 25 mm. on April 1 reduced the daily rate by about 0s·06. The further reduction by 17 mm. from 27 mm. to 10 mm. only produced a reduction in the rate of about 0s·04. In each case the reduction of pressure resulted in an increased arc and made only a slight alteration in rate.

TABLE I.

Observed Clock Rates (see also Table V.).

	Daily Losing Rate. s	Change in a Month. s
1925 January	+·027	
		+·041
February	+·068	
		+·038
March	+·106	
April	+·081	
		+·031
May	+·112	
		+·037
June	+·149	
July	+·145	
		+·046
August	+·191	
		+·038
September	+·229	
		+·015
October	+·244	
		+·014
November	+·258	
		+·010
December	+·268	
		+·007
1926 January	+·275	
		+·012
February	+·287	
		+·012
March	+·299	
		+·009
April	+·308	
		+·017
May	+·325	

It will be seen later that when allowance is made for the effect of temperature the change in the rate is more regular, part also of the change of rate was due to leak in the case.

The clock stopped on 1926 June 10 as a result of the failure of the release of the lever which resets the impulse arm of the free pendulum. The failure was caused by two steel surfaces jamming. To avoid recurrence of this fault a jewel-locking pallet was introduced and some minor alterations were made before the clock was restarted.

From 1926 July.—It was then found that the rate was increasing at nearly the same rate as it had been before the stoppage; viz. about

$^{s}\cdot011$ per day per month. When allowance was made for this change the observed clock errors were satisfactorily explained. Since resealing there has been no leak in the case, the pressure remaining constant at 35 mm. During 1926 October and November special time observations were made in connection with the longitude determinations which were being carried out in many countries under the auspices of the International Astronomical Union. As it happened, the temperature in the clock room began to fall fairly rapidly in the second half of October, and on November 11 electric heating controlled by thermostat was introduced, and till the summer of 1927 the mean temperature was kept near 55° F., although the control of the thermostat was not within narrow limits. The changes of rate which took place in 1926 October and November appeared to be explicable by the old secular term and a temperature term. Although it was not easy to isolate these two effects it appeared that a rise of 1° F. increased the daily losing rate by about $^{s}\cdot002$.

From 1926 November till 1927 May the increase of rate was very steady. In 1927 May, however, the error started to depart from the predicted value, and on investigation it was found that the difference could be completely explained by the irregularity in sidereal time due to nutation. The observations made with the transit circle from 1926 December 1 to 1927 April 30 were analysed in groups of about a week each.

The analysis gave the following formulæ for the clock correction, t being reckoned in units of 100 days from 1927 February $10^{d}\ 0^{h}$:—

$$(1)\quad -3^{s}\cdot810 - 1^{s}\cdot895t + 1^{s}\cdot294t^{2}$$

when no account was taken of the irregularity due to nutation, and

$$(2)\quad -2^{s}\cdot820 - 1^{s}\cdot866t + 1^{s}\cdot535t^{2} + \text{Nutation}$$

when nutation was allowed for.

The following table gives for intervals of 10 days from 1926 November 26 to 1927 July 14 the value of (1), (2) and the observed clock correction with various differences. For this table the clock error was read from a curve drawn through the observed values.

TABLE II.

Observed and Computed Clock Errors, 1926 Dec.–1927 July.

	(1).	(2).	Obs.	Obs.–(1).	Obs.–(2).
	s	s	s	s	s
1926 Nov. 26	−1·466	−1·458	−1·400	+·066	+·058
Dec. 6	1·849	1·855	1·813	+·036	+·042
16	2·207	2·215	2·182	+·025	+·033
26	2 539	2·543	2·585	−·046	−·042
1927 Jan. 5	2·845	2·841	2·890	−·045	−·049
15	3·125	3·112	3·160	−·035	−·048
25	−3·379	−3·361	−3·400	−·021	−·039

TABLE II.—*continued.*

	(1).	(2).	Obs.	Obs.−(1).	Obs.−(2).
	s	s	s	s	s
1927 Feb. 4	−3·607	−3·587	−3·552	+·055	+·035
14	3·810	3·792	3·760	+·050	+·032
24	3·987	3·978	4·000	−·013	−·022
Mar. 6	4·137	4·138	4·120	+·017	+·018
16	4·263	4·273	4·290	−·027	−·017
26	4·361	4·379	4·295	+·066	+ 084
Apr. 5	4·435	4·453	4·420	+·015	+·033
15	4·481	4·492	4·530	−·049	−·038
25	4·503	4·493	4·522	−·019	−·029
May 5	4·498	4·456	4·550	−·052	−·094
15	4·468	4·378	4·420	+·048	−·042
25	4·411	4·264	4·300	+·111	−·036
June 4	4·329	4·115	4·100	+·229	+·015
14	4·221	3·920	3·870	+·351	+·050
24	4·087	3·699	3·600	+·487	+·099
July 4	3·927	3·446	3·305	+·622	+·141
14	−3·742	−3·167	−3·050	+·692	+·117

In the above table the values are all given to $^{s}\cdot001$, but the observed values are liable to errors of the order of $\pm\,^{s}\cdot04$. The only observations between February 10 and February 23 were three clock stars without azimuth stars, and consequently the curve had to be drawn somewhat arbitrarily. The irregularities may be at least partly due to the observations. As stated above, the observations used in deriving the formulæ (1) and (2) only extended to April 30. By May 25 it will be seen that the formula which did not take account of nutation was in error by $0^{s}\cdot1$, and that thereafter the error rapidly increased. By the end of June the temperature in the clock cellar had risen 2° F. above that at which it had been controlled in the early part of the year, and it became obvious that there was a temperature term in the rate of the clock.

On account of the excellent performance of the clock, special time determinations were undertaken with the small reversible transit B, fitted with a self-registering micrometer, which had been used in the longitude work of 1926 October and November. The instrument was reversed on each star and the observations were carried over about 5 revolutions of the screw in each position, each revolution giving three contacts. The stars were generally observed symmetrically, so that not only collimation but also screw value, error of screw, optical distortion and irregular illumination, if any, were eliminated. Instead of the Greenwich star places which were used in reducing observations with the transit circle all the observations were reduced by means of the star places given by Eichelberger and now in use in the national ephemerides. A large number of levels were taken, and each observa-

tion was read to $^{s}\cdot01$ from the chronograph sheets. The short-period lunar terms were allowed for in deriving the star places.

Meanwhile the temperature in the clock room rose to 61° F. at the beginning of 1927 September, and it was not till December that it fell to 55°, when it was controlled again near that temperature. All the clock corrections found with the small transit from 1927 March 17 to 1927 December 30 were then analysed in the form

$$\alpha + \beta t + \gamma t^2 + \delta\int(\mathrm{T} - 54^\circ\cdot8)dt + \text{Nutation},$$

where t is measured in intervals of 100 days from 1927 December $1^{d}\ 0^{h}$, and the lower limit of the integral is for that date.

In the analysis it was found that β, γ, δ were not very easily separated from one another. The coefficients of t and t^2 in (3) were found from the rates when the temperature was nearly constant, and the temperature coefficient was then derived by comparison of residuals with the temperature integral.

$$(3)\quad 7^{s}\cdot775 + 8^{s}\cdot3636t + 1^{s}\cdot840t^2 + 0^{s}\cdot298\int(\mathrm{T} - 54^\circ\cdot8)dt + \text{Nutation}.$$

The residuals between the observed values and those deduced from this formula are given in $\mathrm{O}-\mathrm{C}_3$ in Table III. A rough solution of the residuals $\mathrm{O}-\mathrm{C}_3$ leads to the formula

$$(3)'\quad 7^{s}\cdot748 + 8^{s}\cdot3510t + 1^{s}\cdot8482t^2 + 0^{s}\cdot3047\int(\mathrm{T} - 54^\circ\cdot8)dt + \text{Nutation}.$$

The residuals from this formula are given in the last column of Table III. under the heading $\mathrm{O}\ \mathrm{C}_3'$. The two sets of figures in the column headed No. of Stars refer to clock stars and azimuth stars respectively.

TABLE III.

Comparison of Clock Errors Observed with Transit B with those Computed from Formulæ (3) and (3)′.

1927.		$\int(\mathrm{T}-54^\circ\cdot8)dt$.	$^{s}\cdot298\int$.	Comp. (3).	Obs.	$\mathrm{O}-\mathrm{C}_3$.	No. of Stars.	$\mathrm{O}-\mathrm{C}_3'$.
		°	s	s	s	s		s
Mar.	17	−6·303	−1·878	−4·461	−4 472	−·011	4, 4	−·028
	20	−6·304	−1·879	−4·519	−4·519	·000	5, 3	−·016
May	24	−6·268	−1·868	−4·446	−4·413	+·033	6, 4	+·050
	28	−6·238	−1·859	−4·393	−4·331	+·062	6, 3	+·079
June	12	−6·103	−1·819	−4·055	−4·065	−·010	8, 6	+·013
July	19	−5·748	−1·713	−2·837	−2·906	−·069	6, 6	−·036
	21	−5·705	−1·700	−2·760	−2·821	−·061	4, 4	−·026
	28	−5·541	−1·651	−2·427	−2·522	−·095	4, 3	−·059
Aug.	8	−5·105	−1·521	−1·881	−1·928	−·047	6, 3	−·010
	9	−5·055	−1·506	−1·820	−1·934	−·114	2, 3	−·078
	18	−4·563	−1·360	−1·292	−1·273	+·019	2, 3	+·056
	25	−4·125	−1·229	−0·818	−0·850	−·032	4, 4	+·004
Sept.	7	−3·360	−1·001	+0·031	+0·051	+·020	5, 4	+·055
	12	−3·041	− ·906	+0·376	+0·420	+·044	3, 1	+·076

TABLE III.—*continued.*

1927.	$\int(T-54°{\cdot}8)dt.$	$^{s}{\cdot}298\int.$	Comp. (3).	Obs.	O−C$_3$.	No. of Stars.	O−C$_3$′.
	°	s	s	s	s		s
Sept. 15	−2·853	− ·850	+0·580	+0·550	−·030	1, 1	+·001
17	−2·733	− ·814	+0·732	+0·718	−·014	4, 3	+·018
24	−2·331	− ·695	+1·245	+1·211	−·034	1, 1	−·004
Oct. 3	−1·869	− ·557	+1·891	+1·793	−·098	5, 4	−·068
5	−1·776	− ·529	+2·054	+1·984	− 070	3, 2	−·040
10	−1·541	− ·459	+2·429	+2·296	−·133	4, 4	−·104
18	−1·218	− ·363	+3·012	+2·858	−·154	4, 2	−·127
28	− ·895	− ·267	+3·798	+3·710	−·088	1, 0	−·061
31	− ·791	− ·236	+4·060	+4·063	+·003	4, 4	+·030
Nov. 4	− ·631	− ·188	+4·412	+4·394	−·018	4, 3	+·009
9	− ·430	− ·128	+4·819	+4·814	−·005	5, 4	+·022
16	− ·270	− ·080	+5·444	+5·439	−·005	4, 4	+·022
Dec. 6	+ ·061	+ ·018	+7·191	+7·184	−·007	4, 4	+·020
15	+ ·067	+ ·020	+8·042	+8·057	+·015	3, 0	+·044
16	+ ·058	+ ·017	+8·112	+8·118	+·006	5, 4	+·034
28	+ ·089	+ ·026	+9·264	+9·298	+·034	6, 5	+·064
30	+ ·098	+ ·029	+9·450	+9·417	−·033	4, 2	−·004

(For continuation see Addendum, p. 481.)

The temperature correction was derived as follows. A thermograph had been installed in the clock room on 1927 February 8. From the trace the mean temperature was read by mere inspection to the nearest tenth of a degree. The integrals were obtained by summing these daily means and dividing by 100 to change to units of 100 days. No account was taken of fractions of a day. Thus the difference of $\int(T - 54°{\cdot}8)dt$ between March 17 and March 20 is explained by the temperatures 55°·1, 54°·8, 54°·4 read off for March 18, 19, 20.

On account of the success with which the observations had been explained between 1927 March 17 and December 30 by the formula, it appeared worth while carrying it backwards to test how closely it fitted the transit circle observations from 1926 July onwards; *i.e.* for 450 days in all.

For this purpose it was now necessary to supply temperatures for the earlier results. A maximum and minimum thermometer had been read pretty regularly every day, though in 1926 July there was a week when the temperatures were unreliable.

A comparison of the mean of the maximum and minimum readings with thermograph readings from 1927 February 14 to 1927 July 4 showed that if $\frac{M+m}{2}$ were reduced by °·34 F., the resulting temperatures agreed closely with the thermograph readings. It was thus possible to use the readings of the maximum and minimum thermometer

to carry the comparison backwards. Several corrections have to be applied to the formula (3) to make it suitable for the transit circle observations ; viz.—

(1) $+ \cdot 02^{s}$ for difference of longitude.
(2) $+ \cdot 05$ for difference of star places.
(3) $+ \cdot 03$ for difference of method of reading chronograph sheets, which was relative to beginning of clock signals for small instrument and middle of signals for large instrument.
(4) $+ \cdot 09$ a systematic difference between the two instruments.

Hence for comparison with the transit observations we have the formula, based on (3)—

$$(4)\quad 7^{s}\cdot 965 + 8^{s}\cdot 3636t + 1^{s}\cdot 840t^{2} + {}^{s}\cdot 298\int(T - 54^{\circ}\cdot 8)dt + \text{Nutation},$$

where the zero date is 1927 December $1^{d}\ 0^{h}$.

The comparison of this formula with the observations is given in Table IV., the clock errors having been read off a curve at 10-day intervals. The formula based on (3)′ differs similarly from the observations.

TABLE IV.

Comparison of Clock Errors Observed with the Transit Circle with those Calculated by Formula (4). 1926 *July* 9 *to* 1927 *December* 31.

1926 / 1927	Temp. Corr.	Nut.	$\alpha+\beta t+\gamma t^{2}$.	Comp.	Obs.	O−C.
	s	s	s	s	s	s
July 9	−3·384	−0·978	+13·169	+8·807	+9·010	+·203
19	3·238	0·963	12·147	7·946	7·937	−·009
29	3·047	0·955	11·162	7·160	6·935	−·225
Aug. 8	2·885	0·954	10·214	6·375	5·980	−·395
18	2·736	0·961	9·302	5·605	5·125	−·480
28	2·567	0·977	8·427	4·883	4·290	−·593
Sept. 7	2·388	0·999	7·589	4·202	3·545	−·657
17	2·188	1·025	6·788	3·575	2·907	−·668
27	2·008	1·054	6·024	2·962	2·225	−·737
Oct. 7	1·865	1·081	5·296	2·350	1·597	−·753
17	1·738	1·104	4·604	1·762	1·000	−·762
27	1·713	1·120	3·950	1·117	+0·370	−·747
Nov. 6	1·787	1·128	3·333	+0·418	−0·333	−·751
16	1·840	1·125	2·753	−0·212	0·925	−·713
26	1·856	1·113	2·210	0·759	1·400	−·641
Dec. 6	1·867	1·093	1·703	1·257	1·813	−·556
16	1·872	1·068	1·233	1·707	2·182	−·475
26	1·864	1·040	0·799	2·105	2·585	−·480
Jan. 5	1·875	1·013	0·403	2·485	2·890	−·405
15	−1·874	−0·990	+0·043	−2·821	−3·160	−·339

TABLE IV.—*continued.*

1927.	Temp. Corr.	Nut.	$\alpha+\beta t+\gamma t^2$.	Comp.	Obs.	O−C.
	s	s	s	s	s	s
Jan. 25	−1·861	−0·975	−0·280	−3·116	−3·400	−·284
Feb. 4	1·859	0·969	0·566	3·394	3·552	−·158
14	1·862	0·972	0·815	3·649	3·760	−·111
24	1·881	0·986	1·027	3·894	4·000	−·106
Mar. 6	1·876	1·006	1·203	4·085	4·120	−·035
16	1·885	1·031	1·342	4·258	4·290	−·032
26	1·889	1·059	1·444	4·392	4·295	+·097
Apr. 5	1·894	1·084	1·509	4·487	4·420	+·067
15	1·899	1·105	1·537	4·541	4·530	+·011
25	1·888	1·119	1·529	4·536	4·522	+·014
May 5	1·880	1·125	1·485	4·490	4·550	−·060
15	1·887	1·122	1·403	4·412	4·420	−·008
25	1·875	1·111	1·284	4·270	4·300	−·030
June 4	1·847	1·094	1·128	4·069	4·100	−·031
14	1·817	1·071	0·935	3·823	3·870	−·047
24	1·785	1·047	0·707	3·539	3·600	−·061
July 4	1·779	1·023	0·441	3·243	3·305	−·062
14	1·753	1·002	−0·138	2·893	3·050	−·157
24	1·688	0·987	+0·202	2·473	2·630	−·157
Aug. 3	1·601	0·978	0·579	2·000	2·170	−·170
13	1·461	0·978	0·991	1·448	1·587	−·139
23	1·286	0·986	1·441	0·831	0·885	−·054
Sept. 2	1·111	1·002	1·928	−0·185	−0·270	−·085
12	0·926	1·022	2·452	+0·504	+0·470	−·034
22	0·745	1·047	3·013	1·221	1·145	−·076
Oct. 2	0·686	1·072	3·609	1·851	1·853	+·002
12	0·446	1·094	4·243	2·703	2·540	−·163
22	0·332	1·111	4·914	3·471	3·343	−·128
Nov. 1	0·236	1·120	5·622	4·266	4·230	−·036
11	0·119	1·119	6·366	5·128	5·080	−·048
21	−0·055	1·109	7·147	5·983	5·985	+·002
Dec. 1	0·000	1·090	7·965	6·875	6·892	+·017
11	+0·024	1·063	8·820	7·781	7·824	+·043
21	0·015	1·033	9·712	8·694	8·754	+·060
31	+0·029	−1·002	+10·640	+9·667	+9·708	+·041

The residuals obtained show that for the eleven months 1927 February to December the formula explained the observed clock corrections to within $\pm 0^s{\cdot}1$. The residuals for this interval may be partly explained by errors of observation including possible variations of lag between either the clock and chronograph or the transit instrument and chronograph. It is also quite likely that the temperatures

used are not very accurate. The temperature coefficient used is, however, only $^{s}\cdot00298$ per day per 1° F., or very nearly $1^{s}\cdot1$ per year per 1° F.

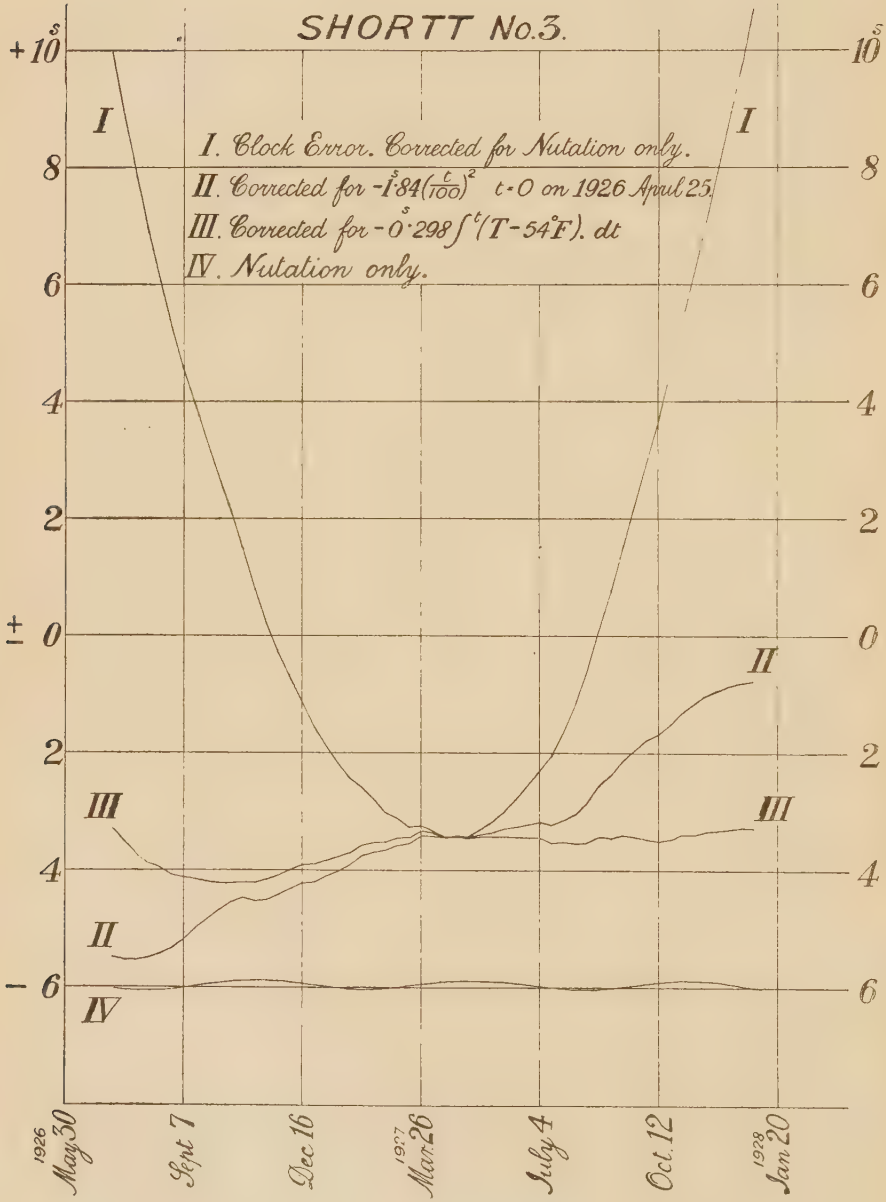
It will be seen that when the formula is extrapolated backwards it agrees with the observed clock error in the middle of 1926 July, but that in 1926 October it is three-quarters of a second in error. A comparison of the values O−C with the run of the temperature corrections makes it appear unlikely that uncertainty in the temperature correction is responsible. It appears more likely that the pendulum rod was growing irregularly in the second half of 1926. It is, in fact, a known property of invar that its length is subject to uncertain variation.

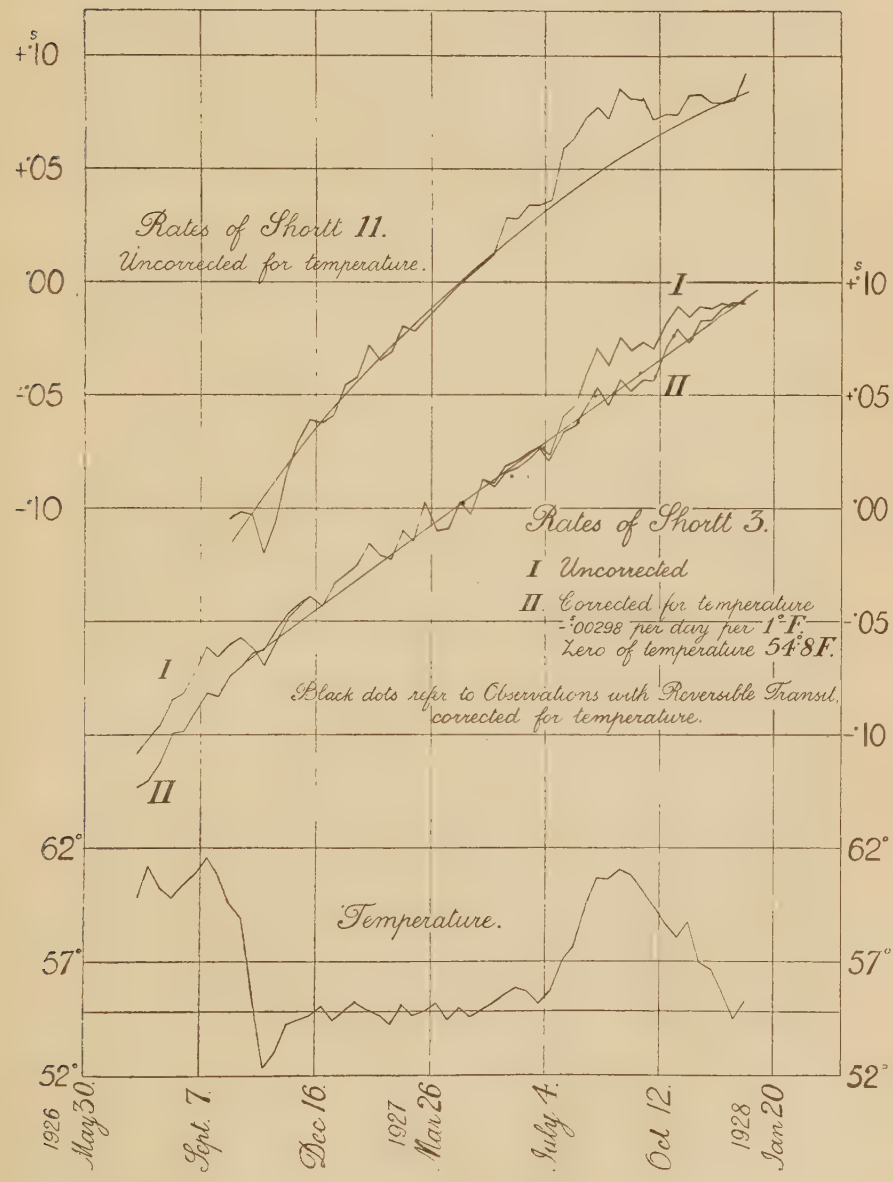
We now return to the behaviour of the clock from 1925 January to 1926 May. To the results given in Table I. we have now to apply corrections for nutation and corrections for temperature. The results are given in Table V.

TABLE V.

Corrected Rates, 1925 *January to* 1926 *May.*

	Observed Rate.	Corrections for		Corrected Rate.	Change per Month.	Pressure in Inches.	Change per Month.
		Nutation.	Temp.				
		In Units of $^{s}\cdot001$.					
	s			s	s		
1925 Jan.	+·027	−2	− 1	+·024		1·220	
					+·043		+·056
Feb.	+·068	+1	− 2	+·067		1·276	
					+·042		+·055
Mar.	+·106	+3	0	+·109		1·331	
					−·031		−·291
Apr.	+·081	+2	− 5	+·078		1·040	
					+·029		+·012
May	+·112	−1	− 4	+·107		1·052	
					+·026		+·025
June	+·149	−3	−13	+·133		1·077	
					−·008		−·643
July	+·145	−2	−18	+·125		0·434	
					+·046		+·033
Aug.	+·191	+1	−21	+·171		0·467	
					−·044		+·011
Sept.	+·229	+3	−17	+·215		0·478	
					+·021		+·036
Oct.	+·244	+2	−10	+·236		0·514	
					+·021		+·018
Nov.	+·258	−1	0	+·257		0·532	
					+·014		+·024
Dec.	+·268	−3	+ 6	+·271		0·556	
					+·010		+·036
1926 Jan.	+·275	−2	+ 8	+·281		0·592	
					+·012		+·032
Feb.	+·287	+1	+ 5	+·293		0·624	
					+·017		+·047
Mar.	+·299	+3	+ 8	+·310		0·671	
					+·007		+·026
Apr.	+·308	+2	+ 7	+·317		0·697	
					+·015		+·053
May	+·325	−1	+ 8	+·332		0·750	


SHORTT No.3.
+10s
8
6
4
2
± 0
2
4
− 6
10s
8
6
4
2
0
2
4
6
I
I
I. Clock Error. Corrected for Nutation only.
II. Corrected for -1s84(t/100)² t=0 on 1926 April 25
III. Corrected for -0s·298 ∫t(T-54°F). dt
IV. Nutation only.
II
III
III
II
IV
1926 May 30
Sept 7
Dec 16
1927 Mar 26
July 4
Oct 12
1928 Jan 20


s
+·10
+·05
·00
-·05
-·10
Rates of Shortt 11.
Uncorrected for temperature.
I
II
Rates of Shortt 3.
I Uncorrected
II Corrected for temperature
-·00298 per day per 1°F.
Zero of temperature 54°8F.
Black dots refer to Observations with Reversible Transit, corrected for temperature.
62°
57°
52°
Temperature.
1926 May 30.
Sept. 7.
Dec. 16.
1927 Mar. 26.
July 4.
Oct. 12.
1928 Jan. 20.

The rates given in Table V. cannot be in error by more than a few units in the last decimal place given. It will be seen that the irregularity in the first six months might be fairly well attributed to leak in the case together with a change in the rate of $^{s}\cdot 030$ per day per month. The change in the acceleration after September is, however, unexplained.

In pointing out the fluctuations in the rate of the clock it is only right to add that the clock has been subjected to treatment from which considerable fluctuations might have been expected. We have already referred to the changes which have taken place in the temperature. Alterations were made in the clock cellar as follows :—

(1) 1925 August, an additional "rendering" was put on some of the walls and floor, and some heavy stone shelving removed.
(2) 1926 February, the south wall of the cellar was bored in preparation for mounting Clock No. 11. For this purpose four holes $\frac{7}{8}$-inch diameter were drilled through a wall 3 ft. 7 in. thick.
(3) Following this a hole 4 in. in diameter and 3 ft. 10 in. in length was bored through the north wall for ventilation.

Some four weeks in all were spent on the above alterations, during which it is considered that the clock was under abnormal conditions.

Clock Shortt 11.—This clock was delivered at the Observatory on 1926 May 20. The free pendulum was set up in the same vault as No. 3 but so as to swing in a plane at right angles to No. 3. The slave clock was set up beside that of No. 3. Comparisons between the two clocks were made twice daily. Table VI. gives the mean difference of the two clocks and the rates of No. 11 on No. 3, and then, by means of the observed correction to No. 3, the deduced correction of No. 11 on mean sidereal time and its rate. On 1926 September 21 the synchronising circuit of No. 11 was inadvertently broken. Between 1927 March 22 and 1927 May 9 the second-beats of the two clocks were so near together that it was impossible to read them on the chronograph. From May 10 to May 28 the comparisons were made by means of a third clock. It will be seen that the rate of No. 11 had a temperature coefficient nearly equal to that of No. 3—a point which is shown clearly on the diagram showing the rates and the temperature. Throughout the whole of 1927 the losing rate of No. 3 was increasing nearly uniformly, while the rate of increase of losing rate of No. 11 was falling off slowly. It will be seen that the temperature coefficient of No. 11 was nearly the same as (or perhaps a little larger than) that of No. 3, although it cannot be easily disentangled from the secular change of rate.

TABLE VI.

Comparison of Shortt 11 *with Shortt* 3 *and with Mean Sidereal Time.*

1926/1927.	No. 3 Minus No. 11.	Relative 10-day Rate.	Corr. of No. 3 to Mean Sid. Time.	Corr. of No. 11 to Mean Sid. Time.	10-day Rate of No. 11.
	s	s	s	s	s
Aug. 8	+2·456		+6·934	+9·390	
		−·352			−1·200
18	2·104		6·086	8·190	
		·353			1·172
28	1·751		5·267	7·018	
		·414			1·137
Sept. 7	1·337		4·544	5·881	
		·354			0·966
17	0·983		3·932	4·915	
		·468			1·121
27	0·515		3·279	3·794	
		·446			1·047
Oct. 7	+0·069		2·678	2·747	
		·441			1·015
17	−0·372		2·104	1·732	
		·411			1·025
27	0·783		1·490	+0·707	
		·502			1·197
Nov. 6	1·285		0·795	−0·490	
		·461			1·056
16	1·746		+0·200	1·546	
		·356			0·843
26	2·102		−0·287	2·389	
		·271			·704
Dec. 6	2·373		0·720	3·093	
		·211			·605
16	2·584		1·114	3·698	
		·189			·620
26	2·773		1·545	4·318	
		·266			·598
Jan. 5	3·039		1·877	4·916	
		·150			·444
15	3·189		2·171	5·360	
		·171			·425
25	3·360		2·425	5·785	
		·122			·280
Feb. 4	3·482		2·583	6·065	
		·141			·345
14	3·623		2·787	6·410	
		·083			·310
24	3·706		3·014	6·720	
		·097			·197
Mar. 6	3·803		3·114	6·917	
		·069			·214
16	3·872		3·259	7·131	
26	..		3·236		
Apr. 5	..		3·336		
15	..	−·019	3·426	..	− ·025
25	..		3·403		
May 5	..		3·425		
15	−3·985		−3·298	−7·283	
		+·017			+ ·126

TABLE VI.—*continued.*

1927.	No. 3 Minus No. 11.	Relative 10-day Rate.	Corr. of No. 3 to Mean Sid. Time.	Corr. of No. 11 to Mean Sid. Time.	10-day Rate of No. 11.
	s	s	s	s	s
May 25	−3·968		−3·189	−7·157	
		+·100			+·282
June 4	3·868		3·007	6·875	
		·072			·280
14	3·796		2·799	6·595	
		·100			·345
24	3·696		2·554	6·250	
		·074			·345
July 4	3·622		2·283	5·905	
		·128			·363
14	3·494		2·048	5·542	
		·188			·593
24	3·306		1·643	4·949	
		·186			·637
Aug. 3	3·120		1·192	4·312	
		·140			·723
13	2·980		−0·609	3·589	
		·063			·773
23	2·917		+0·101	2·816	
		·098			·729
Sept. 2	2·819		0·732	2·087	
		·101			·861
12	2·718		1·492	1·226	
		·112			·812
22	2·606		2·192	−0·414	
		·086			·819
Oct. 2	2·520		2·925	+0·405	
		+·014			·723
12	2·506		3·634	1·128	
		−·073			·747
22	2·579		4·454	1·875	
		·152			·744
Nov. 1	2·731		5·350	2·619	
		·019			·830
11	2·750		6·199	3·449	
		·058			·837
21	2·808		7·094	4·286	
		·089			·799
Dec. 1	2·897		7·982	5·085	
		·109			·796
11	3·006		8·887	5·881	
		−·096			·804
21	3·102		9·787	6·685	
		·000			+·923
31	−3·102		+10·710	+7·608	

Conclusion.—It will be seen that during the year 1927 the principal irregularities in the going of the two Shortt clocks can be explained by a secular change in the period of oscillation and a temperature effect. The latter is about ^{s}·003 per day per 1° F. or, say, 1^{s}·1 per year per 1° F. This is comparatively small and it should be easily possible to control the temperature with sufficient accuracy for all purposes. This temperature term is small compared with that of most clocks, but it had not been expected to exist in the Shortt clocks. It appears that

only a fraction of it can be attributed to uncertainty in the coefficient of expansion of the materials. It might possibly be due to variation of the arc of oscillation. A variation of arc might arise from a variation of impulse on account of friction. The semi-arc of the Shortt clocks at Greenwich is about 55′. A variation in this of 1′ would, according to the theory of a simple pendulum, produce a change of daily rate of $0^{s}.050$, so that a variation of only 4″ in the semi-arc would suffice to explain the change of rate caused by a change of 1° F.* From the actual performance of the clock it is clear that either the change of arc is very small or the computed alteration of rate with arc is much larger for a simple pendulum than for the actual free pendulum. Unfortunately, it is not now possible to read the arcs of the Greenwich clocks to test this point, although such readings as have been made indicate that the arc might vary through 30″. Whenever the clock cases are opened means will be provided for reading the arc accurately, but it is not intended at present to interfere with the pendulums, at least not that of No. 3. It has also been suggested that the material of the suspension spring may be responsible for some irregularity in the clock.

A comparison of the residuals of Tables III. and IV. shows that the formula does not exactly fit the observations. Also, as there were four constants available, the results might explain away irregularities in the clock errors. The attempt to compute the observed errors for the second half of 1926 from the errors observed between 1927 March and December gave systematic residuals of $0^{s}.75$. These residuals give a clear proof that the secular change of the rate was not uniform. In fact, formula (2) shows a secular term of only $1^{s}.535t^{2}$ from the observations of 1926 December to 1927 April as compared with $1^{s}.840t^{2}$ from 1927 March to 1927 December. From 1926 July to 1926 September the secular term was larger. The comparison of the two clocks given in Table VI. does not involve any astronomical observations, and shows that the relative change of rates was not linear. If we take the rates for 40-day intervals at dates 90 days apart we find :—

	No. 3 Losing on No. 11.	Difference.
	s	s
1926 Aug. 28	−·0368	
		+·0043
Nov. 26	−·0325	
		+·0227
1927 Feb. 24	−·0098	
		+·0156
May 25	+·0058	
		+·0042
Aug. 23	+·0100	
		−·0169
Nov. 21	−·0069	

* The observations so far as they go, do not, however, support this hypothesis. With the temperature 6° F. above normal the rate in the summer was $^{s}.018$ in excess of its computed value. This corresponds to a change of arc of 20″, just about the observable limit. Actually the arc at the time of highest temperature appeared to be rather less than it had been two months earlier.

It appears from the table that the acceleration of the one clock on the other was not uniform. The errors of the comparison can only amount to 1 or 2 units in the last decimal place. Part of the irregularity may easily be due to difference in the temperature coefficient of the two clocks, that of No. 11 being apparently greater than that of No. 3.

From the figures given it will be seen that the performance of these clocks far surpasses that of any other type. When we consider the mechanical disturbance to which they have been liable during alterations in the clock room the uniformity of the rate has been wonderful. The variations of temperature to which they have been subjected have shown the existence of a temperature term. These are conditions to which a clock will not be subjected when an attempt is being made to give it a fair chance of showing its best.

The construction of the clocks appears to justify the name "free pendulum." The only mechanical interference with the oscillation of the pendulum takes place when the impulse is given. This lasts for part of a second when the pendulum moves through the zero position at every fifteenth swing. There appears to be little alteration in the going of the clock due to variation in the impulse. The remaining possibilities for irregularity are in :—

(1) The length of the rod.
(2) The suspension spring.
(3) Gravity.
(4) The rotation of the earth.
(5) Earth tremors.

Of these (1) is the more important. If allowance is made for a steady growth in the rod and for the temperature term throughout 1927 the remaining irregularities of No. 3 appear to be less than 1 part in 100 millions.

The question has been raised as to the possibility of using the Shortt clocks to check the uniformity of the earth's rotation. Observations of the position of the moon, supported fairly well by observations of more slowly moving bodies in the solar system, indicate that the period of rotation of the earth may be subject to variation amounting in extreme cases to a change of 1^s per year. It appears from the performance of the clocks that the invar rod is far too unstable to test such a change. The term $1^s{\cdot}84t^2$ in the clock error would amount to over 3 minutes in 3 years. This term can be explained by a growth of the invar rod of about 1 micron (one-thousandth of a millimetre) in 120 days. Although this growth has been maintained at a uniform rate throughout 1927 for No. 3, it appears to have been greater at the beginning of the year and less at the end for No. 11. A variation in the growth of 2 per cent. per annum of that found for Shortt 3 would make a pendulum of invar unsatisfactory for testing the earth's rotation. It appears from the behaviour of these clocks that there might be some prospect of success if a stable material with a small temperature coefficient could be obtained for the construction of the pendulum rod.

Addendum.—It is interesting to compare the clock errors derived from observations with Transit B in 1928 with formulæ (3) and (3)′. The continuation of Table III. runs as follows ; the clock dial has been altered 10^s.

1928.	$\int(T-54°\cdot8)dt$.	$s\cdot298\int$.	Comp. (3).	Obs.	O−C₃.	No. of Stars.	O−C₃′.
	°	s	s	s	s		s
Jan. 6	+·117	+·035	+0·179	+0·176	−·003	4, 2	+·026
19	+·170	+·051	+1·488	+1·482	−·006	6, 4	+·024
Feb. 1	+·199	+·059	+2·868	+2·838	−·030	6, 4	·000
13	+·236	+·070	+4·179	+4·102	−·077	6, 4	−·047
20	+·258	+·077	+4·993	+4·897	−·096	5, 3	−·066
28	+·283	+·084	+5·890	+5·804	−·086	3, 3	−·056

Preliminary Values of the Variation of Latitude at Greenwich in 1927.

(*Communicated by the Astronomer Royal.*)

This paper contains the preliminary values of the latitude variation derived from observations made during 1927 with the Cookson Floating Telescope.

The star places used were those derived from the final discussion of the observations for the years 1919–26. Each Talcott pair has been treated as a separate unit, corrections for proper motion and temperature having been applied to each pair instead of to each plate (containing 4 or 5 pairs) as formerly.

The variation of latitude has again been very small, as was the case in the two previous years. The following table gives the values of the latitude variation, in the sense mean latitude *minus* observed :—

Date.	Variation.	Number of Pairs.
	″	
1927·044	−·121	78
·092	−·080	16
·186	−·045	92
·264	−·015	72
·350	−·018	92
·498	−·011	51
·585	−·057	72
·676	−·071	68
·771	−·053	117
·909	+·083	78
1928·033	+·026	109

From a smooth curve drawn to represent the observed results as closely as possible, the following corrections, applicable to observed north polar distances, were derived :—

1927·0	−″·10	1927·6	−″·06
27·1	−·06	27·7	−·09
27·2	−·04	27·8	−·04
27·3	−·02	27·9	·00
27·4	−·01	1928·0	+·06
27·5	−·03		

The final reduction of the observations from 1919 to 1926 will be published shortly.

Royal Observatory, Greenwich :
March 1928.

Perturbations in the System of β Lyræ. By S. D. Tscherny.

(*Communicated by the Secretaries.*)

The consideration of my paper (*A.N.*, Bd. 230, *Nr.* 5504) at the R.A.S. Meeting in January, in connection with Miss Blagg's paper on β Lyræ (as reported in *The Observatory* for February), gives me occasion to make some additional remarks on this paper.

1. The periodic perturbations of the longitude of the periastron are :

(*a*) $\frac{3\mu_2}{n^2a^5e}\sin nt$ with large amplitude ($e = 0{\cdot}007$) and the period T of the light variation of β Lyræ.

(*b*) $\frac{3\mu_2}{n^2a^5}\sin 2nt$ with small amplitude and period $\frac{T}{2}$.

Putting together these periodic perturbations, we obtain

$$\frac{3\mu_2}{n^2a^5e}(\sin nt + e\sin 2nt) = A\sin\left(\frac{2\pi t}{T+\Delta T}\right) = A\sin\left[\frac{2\pi t}{T}\left(1-\frac{\Delta T}{T}\right)\right],$$

where A is the amplitude of the resulting perturbation and ΔT is a small quantity of the order of e.

With sufficient approximation we have

$$A = \frac{3\mu_2}{n^2a^5e}, \qquad \Delta T = -\frac{2eT}{n}\frac{\sin nt}{t}.$$

The mean value of ΔT is

$$(\Delta T)_{mean} = \pm\frac{4eT}{\pi}\int_0^{\frac{T}{4}}\frac{\sin nt}{t}dt = \pm 2eT = \pm 0{\cdot}181.$$

Therefore the periods of the resulting perturbation are $12^d{\cdot}908 \pm 0{\cdot}181$ and the short periods of the oscillation of the line of apsides are $6{\cdot}454 + 0{\cdot}090 = 6{\cdot}544$ and $6{\cdot}454 - 0{\cdot}090 = 6{\cdot}364$.

Miss Blagg has found the periods $6{\cdot}576 - 6{\cdot}595$, which are very close to the first theoretical period.

2. For the variation of the eccentricity we have the formula

$$e = e_0 + \frac{3\mu_2}{n^2a^5}\cos nt, \qquad \text{where} \qquad \frac{3\mu_2}{n^2a^5} = 0{\cdot}284.$$

For $t = 0$ and $t = \frac{T}{2}$ we have $e = e_0 \pm 0{\cdot}284$.

For $t = \frac{T}{4}$ and $t = \frac{3T}{4}$ we have $e = e_0 = 0{\cdot}007$.

Therefore the eccentricity oscillates during the period T between $0{\cdot}291$ and $-0{\cdot}277$.

3. The orbits of the components of β Lyræ are very complicated curves. The motion in these curves could be represented with close approximation by the motion in the ellipse with the oscillating focus and periastron.

All these results are achieved by the supposition that the noses of the components of β Lyræ are turned towards one another. Such a complicated motion is one of the essential causes of the fluctuations of the light-curve and of the composite appearance in the spectrum of β Lyræ.

Astronomical Observatory, Kiew:
1928 *February* 29.

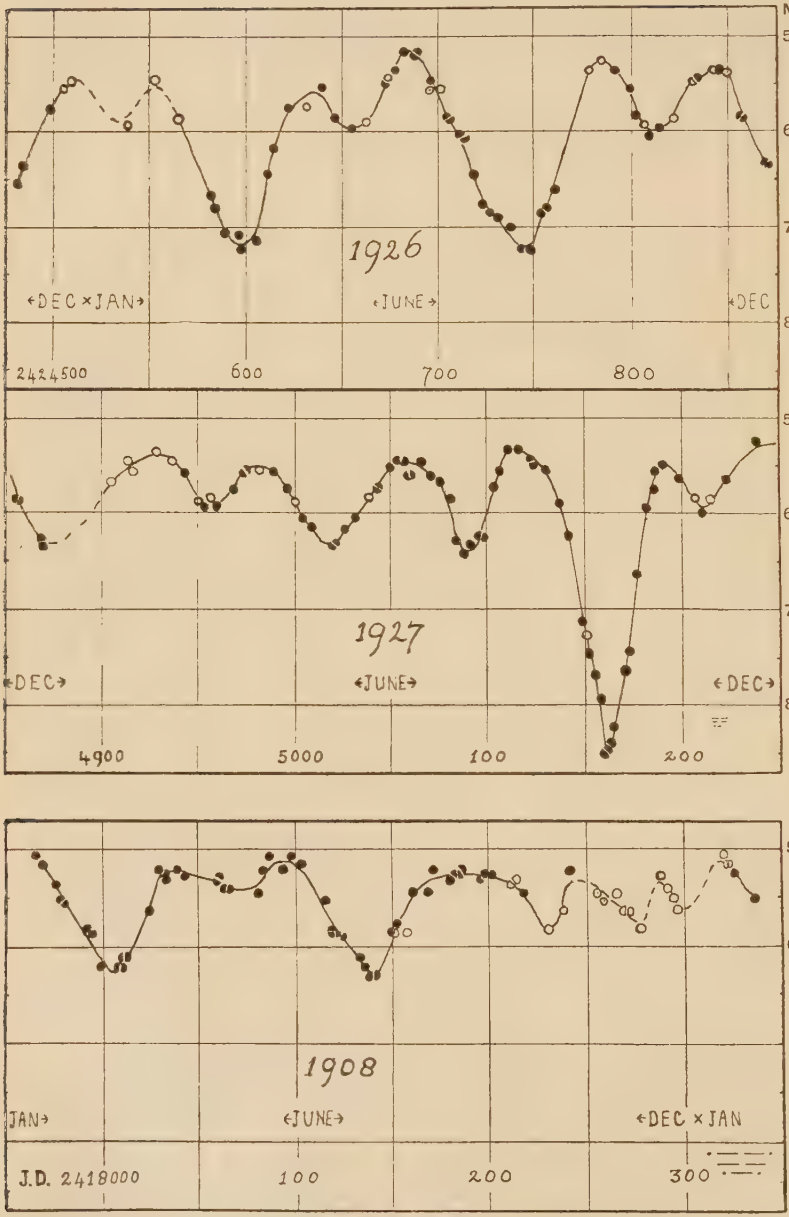
Recent Light-curves of R Scuti. By P. M. Ryves.

R Scuti is a star of special interest, being the brightest of a peculiar group of semi-periodic variables in which the minima are of very unequal depths, the brighter and fainter minima sometimes but not always alternating, whilst the maxima are of more uniform brightness.

Observations of R Scuti have been made by the writer for some years, but have not yet been fully reduced and discussed. The object of the present note is to call attention to certain peculiarities exhibited by the light-curves deduced from the observations of the last two years.

The curve corresponding to 1926 shows three double maxima, of about magnitude 5·4, separated by shallow minima which are all of nearly the same depth (5·9); and two faint minima which are of equal depth (7·2), though the second is somewhat broader. The whole curve is remarkably regular and symmetrical, and the alternation of deep and shallow minima, associated with the variation of RV Tauri, is strictly maintained.

In 1927 the outstanding feature of the variation was the faint minimum which took place in October when, after having remained



R Scuti: Light-curves.

Scale: two magnitudes=100 days=1 English inch (25·4 mm.).

The circles represent second-class observations. The 1908 curve is somewhat less reliable than the later ones. It ought, perhaps, to be lowered a little to compensate a change of habit.

comparatively bright (not below 6·3) for more than eight months, R Scuti fell rapidly to magnitude 8·4, regaining its primitive brightness in an equally short space of time. Such deep minima are rare.

The curves have been smoothed in order to reduce the effect of errors which are obviously present. The observations are most difficult, and consequently most liable to error when the variable is about magnitude 5·5 on account of the lack of convenient comparison stars at that stage; also in the months of December and January when observations are obtained with some difficulty in the evening and morning twilight respectively.

For comparison with the above results the light-curve for 1908 is appended. During this year there were no deep minima, the brightness of the variable never falling below magnitude 6·3. The erratic behaviour of R Scuti at some epochs is no less interesting than its occasional regularity.

Zaragoza, Spain:
1928 *January.*

Occultations of Stars by the Moon, observed at the Tomsk University Observatory during the Year 1927. By Professor N. Goryatscheff.

(*Communicated by the Secretaries.*)

($\phi = 56° 28' 07''$; $\lambda = 5^h 39^m 47^s{\cdot}4$ East of Greenwich.)

Date.	Star.	Mag.	Local Sidereal Time.	Phase.
1927.			h m s	
II. 5	24 B Ceti	6·0	3 31 46·9	D
22	614 B Virginis	6·6	14 27 58·4	R
III. 8	30 B Tauri	6·4	9 59 00·5	D
9	ε Tauri	3·6	8 01 19·2	D
VII. 24	150 B Tauri	6·9	21 21 59·2	R
VIII. 6	32 Libræ	5·9	17 24 18·4	D
7	58 G. Scorpii	6·2	18 24 55·9	D
18	85 Ceti	6·3	21 03 56·9	R
21	282 B Tauri	6·4	22 17 32·2	R
24	B.D.+23°·1744	6·4	0 22 07·0	R
IX. 18	n Tauri	5·1	0 14 55·3	R
19	12 Geminorum	7·0	0 37 21·1	R
19	10 Geminorum	6·6	0 42 49·8	R
19	11 Geminorum	7·0	0 53 34·1	R
19	Berl. B 2314	8·6	1 09 31·2	R
19	Berl. B 2327	8·3	1 57 14·2	R
19	Berl. B 2361	8·1	3 42 48·1	R

Date.	Star.	Mag.	Local Sidereal Time.	Phase.
1927.			h m s	
IX. 23	Berl. A 4019	8·9	4 00 10·9	R
X. 15	ι Tauri	4·7	3 39 26·4	R
18	Berl. B 3203	6·7	2 50 26·3	R
18	Berl. B 3204	8·3	2 58 50·6	R
18	Berl. B 3222	8·6	4 10 12·5	R
18	Berl. B 3269	8·5	7 10 59·0	R
XI. 17	42 Leonis	6·1	5 11 12·1	R
XII. 3	24 B Ceti	6·0	23 39 30·8	D

Apparent Places of Occulted Stars of Magnitude >6·5.

Star.	α.	δ.	Authority.
	h m s	° ′ ″	
614 B Virginis	13 56 13·79	− 7 48 28·9	Hedrick, W.Z.C.
150 B Tauri	3 50 18·10	+16 24 16·8	
10 Geminorum	6 14 28·72	+23 37 59·5	
11 Geminorum	6 14 53·93	+23 30 00·8	
12 Geminorum	6 14 57·52	+23 18 19·0	
Berl. B 2314	6 15 42·30	+23 26 12·7	Paris Astrographic.
Berl. B 2327	6 17 23·10	+23 47 48·6	
Berl. B 2361	6 20 52·79	+23 45 05·3	
Berl. B 3203	7 55 23·37	+23 22 58·1	
Berl. B 3204	7 55 27·58	+23 17 10·8	
Berl. B 3222	7 57 54·66	+23 13 38·3	
Berl. B 3269	8 03 32·67	+22 59 38·7	
Berl. A 4019	9 58 33·06	+16 24 16·3	Bordeaux Astr.

Observer.—Professor N. Goryatscheff.

Instruments.—Zeiss telescope with 90 mm. aperture and magnifying power from 90 to 150 and chronometer of Ericsson.

Observations with eye and ear.

Tomsk University Observatory:
1928 *February* 7.

Observation of the Transit of Mercury, 1927 *November* 10, *at Leiston, Suffolk.* By A. F. Bennett.

The concluding phases of the transit were observed in a perfectly clear sky, the definition being fairly good. A solar diagonal and eye-piece giving a magnification of ×240 were used on the 6-inch Cooke equatorial refractor. The "black-drop" phenomenon was very marked.

The contacts were timed by a chronograph watch referred to a marine chronometer, the error of which was checked by wireless time-signals at $22^h\ 0^m$ on November 9, and at $9^h\ 30^m$ on November 10.

The observed times of contacts were as follows :

First appearance of " black drop " effect .	$8^h\ 28^m\ 0^s$
Apparent third contact	$8^h\ 28^m\ 8^s$
Apparent fourth contact	$8^h\ 29^m\ 36^s \pm$

Computed time for third contact (per N.A. formula) was $8^h\ 28^m\ 23^s{\cdot}2$, so that $O - C = -15^s{\cdot}2$.

Due to atmospheric tremor at the sun's limb, the time recorded for the last contact cannot be given the same weight as those for the two earlier phases.

The position of the Observatory is :

Latitude N. . .	$52^\circ\ 12'\ 2''{\cdot}1$
Longitude E. . .	$1^\circ\ 34'\ 30''{\cdot}2$
Altitude . . .	20 metres.

A Meteoric Shower from Skjellerup's Comet, 1927 *k*.
By Issei Yamamoto.

Dr. Crommelin's revised orbital elements * of Skjellerup's daylight comet, 1927 *k*, suggest that the comet passed, in the last ten days of January, very close to the earth's orbit at its descending node. The closest approach was on January 22 with the distance of about five million kilometres.

In the present year the earth will pass this region on June 7, and on that epoch a special meteoric shower should be expected. I have computed, from Crommelin's elements, the following positions for the expected radiant point of these meteors :—

Right ascension $= 42^\circ{\cdot}3$,† Declination $= +41^\circ{\cdot}4$.

A watch is desired. Unfortunately the moon is full on June 3, so that it may affect the observations. The radiant point is near Algol, and rises very early in the morning at that season.

From the fact that the cometary orbit has a high inclination, it is suspected that the duration of the visibility of these meteors will be brief—possibly within a week. But the recent close approach to the sun might have made the meteoric swarm very diffuse about their mean path.

* *I.A.U. Circulaire* 182.

† Dr. Crommelin writes: "I make the R.A. of the radiant to be 122°. There seems to be no doubt that it should be in the second quadrant. My declination agrees pretty well with Yamamoto's."—THE SECRETARIES.

Kyoto University Observatory :
1928 *March* 15.

Note on the Duplicity of Nova Pictoris.
By W. S. Finsen.

Nova Pictoris (R.A. $6^h\ 34^m\ 57{\cdot}2^s$, Dec. $-62°\ 34'\ 33''$, 1925) was discovered by Mr. R. Watson of Beaufort West, and its departure from the usual Nova characteristics was almost at once evident. The rise to maximum was remarkably slow, and the spectral phenomena were so unusual, that Dr. J. Lunt in a letter to Dr. Innes, dated 1925 Sept. 11, asked whether any peculiarity had been noticed here about its telescopic appearance—"particularly any nebulosity or duplicity." The Nova had not at that time been examined with the 26½-inch refractor, but on receipt of Dr. Lunt's letter steps were taken with that object in view, and Dr. van den Bos examined it on two nights.

> "1925 Nov. 15. Not double, definition fairly good.
> "1926 Jan. 16. Nothing peculiar, except strong red pattern round it—disc sharp and stellar. Seeing 3. 420× and 840×."

In a letter to Mr. Wood, dated 1928 Feb. 8, Prof. B. H. Dawson writes as follows :—

> "Dr. Hartman asked me to look up Nova Pictoris and see if it had an appreciable disc. I found one of about 1″·0, but our seeing was not good enough for high power and a really valid measure. Perhaps v. d. Bos or Finsen could get a measure of the diameter of the disc with your large telescope."

Mr. Wood handed me this extract on the morning of Friday, March 23, and as the 26½-inch refractor was allocated to me that evening, I lost no time in examining the Nova. My first impression was that I had made an error in setting the telescope, and that I was in reality looking at a normal close double star in bad definition. A chart showing the Nova and surrounding stars clearly indicated that there was no mistake, but that the Nova had become nebulous, was distinctly oval, and was probably double. On the following evening Dr. van den Bos, to whom I had reported that the Nova was a peculiar object (refraining, however, from any mention of duplicity), examined it and at once pronounced it double. Mr. Wood and Dr. Innes (who had very kindly come in response to a telephone conversation) also concurred in this.

Both components are nebulous. Each is fairly well defined at the edge, gradually increasing in intensity from the circumference to an unmistakable nucleus at the centre. The nuclei are well separated, but the nebulous envelopes coalesce. The colour is deep yellow with a tinge of red, and the difference of magnitudes, if any, is small.

The following observations have been made at the time of writing :—

1928·227 Owing to bad definition, duplicity was somewhat uncertain, but the image was decidedly oval. Major axis 1″·14, in position angle 81°·5. Minor axis 0″·81 (420×). Combined magnitude 7·4, colour 4−5 (φ).

1928·229 $+0^{h}.6$, 78°·5, 0″·48, 840×, Δ $0^{m}.3$, colours 6–6. Would be separated if not nebulous. Stars distinctly nebulous, but no surrounding nebulosity seen (also with power 140×). Total diameter of visible nebulosity in direction 80° : 1″·67 (840×, dark wires), possibly −0″·15 for diameter of wire (B).

	h	°	″	Def.		m	Colours.		
·229	..	67·5	0·48	2–3	840×	Δ 0·5	5–5	(φ)	
·232	+1·5	75·5	0·49	2	840	0·3	5–5	(φ)	
·232	+1·6	86·6	0·60	2	420	0·3	5–5	(φ)	
·235	+0·4	68·3	0·57	3+	420	0·1	..	(φ)	Separated but for blurred character.
·235	+0·6	68·8	0·45	3	840	−0·1	6–6	(φ)	Nuclei well separated, but the whole blurred.
·235	+0·3	66·2	0·46	3+	420	0·1	..	(B)	Nebulous, but distinct minimum of brightness in centre.
·235	+0·5	69·2	0·49	3	840	0·0	6–6	(B)	
·238	+0·8	67·6	0·52	2	420	0·1	6–6	(B)	
·238	+1·2	69·3	0·50	1+	840	..	..	(B)	
·238	+0·7	59·9	0·57	2	420	0·0	6–6	(φ)	Image very blurred, but cannot be far wrong.
·238	+1·1	65·5	0·46	2	840	..	..	(φ)	

Union Observatory, Johannesburg :
1928 *March* 27.

Erratum.

M.N., **87**, 643, paragraph 4, line 3, *for* 7·2 *read* 8·2.